

The GUT of the LIGHT

On the Abelian complexifications of the Euclidean R^n spaces

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Abstract

The new great development in Physics could be related to the excited progress of a new mathematics: ternary theory of numbers, ternary Pithagor theorem and ternary complex analysis, ternary algebras and symmetries, ternary Clifford algebras, ternary differential geometry, theory of the differential wave equations of the higher degree $n > 2$ and etc. Especially, we expect the powerful influence of this progress into the Standard Model (SM) and beyond, into high energy neutrino physics, Gravity and Cosmology. This can give the further development in the understanding of the Lorentz symmetry and matter-antimatter symmetry, the geometrical origin of the gauge symmetries of the Standard Model, of the 3-quark-lepton family and neutrino problems, dark matter and dark energy problems in Cosmology. The new ambient geometry can be related to a new space-time symmetry leading at high energies to generalization of the Special Theory of Relativity. On the modern level the Standard Model and Cosmology has two main problems for the further progress of the phenomenological description of the huge number of the experimental results getting during the last 20-30 years. The first problem is connected to the geometrical picture of the SM, *i.e.* to the understanding do we need some new geometrical representations about our ambient space-time. This problem could directly related to the breaking mechanism of $SU(2) \times U(1)$ gauge symmetry, to the mass quark-lepton mass spectrum. The second problem is connected to the searching for the new symmetries in modern quantum physics and cosmology. This problem is related with the conjecture that the SM of quarks and leptons cannot be solved in terms of binary Lie groups. This problem we formulated as incompleteness of the Standard Model in terms of the binary Lie groups. The development of theory of binary numbers (real, complex, quaternions, octonions) gave the tremendous progress in theory of Lie algebras, Lie symmetries and geometry. This progress excited the huge number of discoveries in high energy physics and cosmology. However the future progress in SM and in Cosmology cannot be done without further progress in theory of new numbers, exotic symmetries and new geometry. We related the future of this development with \mathbb{C}_n numbers, n -algebras ($n \geq 2$) and corresponding geometrical objects. This way was initiated not only by pragmatic ideas but it was also initiated by algebraic classification of CY_m -spaces for all $m=2,3,4,\dots$ and investigation its singularities. In this article we concentrate on the \mathbb{C}_n , $n = 3, 4, \dots$ numbers and $n = 3, 4, \dots$ algebras and explain the difference between Dynkin graphs for Cartan-Lie algebra classification and Berger graphs for

$n = 3, 4, \dots$ algebras, i.e. the appearance of the longest simple roots with ternary Cartan element $B_{ii} = 3, 4, \dots$ comparing with the case of binary algebras in which the Cartan diagonal element is equal 2, *i.e.* $A_{ii} = 2$.

We will discuss the following results

- \mathbb{C}_n complexification of \mathbb{R}^n spaces
- \mathbb{C}_n structure and the invariant surfaces
- \mathbb{C}_n holomorphicity and harmonicity
- The link between \mathbb{C}_n holomorphicity and the origin of spin $1/n$
- New geometry and N-ary algebras/symmetries
- Root system of a new ternary $TU(3)$ algebra
- N-ary Clifford algebras
- Ternary 9-plet and 27-plet number surfaces

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1 Introduction. A little about History

- 1. Euclid geometry and Pithagorean theorem [1],[2];
- 2. Complex numbers, Euler,s formula, complexification of \mathbb{R}^2 , $U(1) = S^1$ [3]
- 3. Hamilton quaternions, octonions and geometry of unit quaternions and octonions, the $SU(2) = S^3$ and $G(2)$ groups [4, 6]
- 4. Lie algebras and Cartan-Killing classification of Lie algebra
- 5. Geometry of symmetric spaces and its application in physics
- 6. Complex numbers in \mathbb{R}^3 space. Appel sphere and ternary generalization of trigonometric functions. [7], [8, 9]
- 7. The $q^n = 1$ generalisation of the complex numbers in \mathbb{R}^n space. [10, 11],[12]
- 8. Ternary quaternions and $TU(3)$ ternary algebra.[13]
- 9. Complex analysis in \mathbb{R}^3 [14]
- 10. Calabi-Yau spaces and its algebraic classification. [15], [16, 17]
- 11. The reflexive number algebra and Berger graphs. Its link to the n-ary Lie algebras. [18]
- 12. The Standard Model problems and new n-ary algebras/symmetries. Searches for a new geometrical objects through the theory of new numbers.

The modern progress in physics of elementary particles is based on the discovery of the Standard Model defined by internal $SU(3) \times SU(2) \times U(1)$ gauge symmetry and external Poincar/'e symmetry. From the point of view of the vacuum structure the SM rests on the old level, and the Higgs mechanism of the breaking the $SU(2) \times U(1)$ vacuum to the $U(1)^{em}$ vacuum does not give any geometrical picture of the primordial vacuum. As the Standard Model comprises three generations of quarks and leptons, the $SU(3) \times SU(2) \times U(1)$

symmetry cannot fix many parameters (about 25) and cannot explain a lot of physical problems. The big number of the parameters inside the SM and our non-understanding of many phenomena like families, Yukawa interactions, fermion mass spectrum, confinement, the nature of neutrino and its mass origin give us a proposal that the symmetry what we saw inside the SM is only a projection of more fundamental bigger symmetry based on the ternary extension of the binary Cartan-Lie symmmtries. There is also an analogy with the Dark matter problem and following this analogy we proposed an existence of some new ternary symmetries in SM.

To understand the ambient geometry of our world with some extra infinite dimensions one can suggest that our visible world (universe) is just a subspace of a space which “invisible” part one can call by bulk. The visibility of such bulk is determined by our understanding of the SM and our possibilities to predict what could happened beyond its. To find an explanation of the small mass of neutrinos in the sea-saw mechanism it was suggested that in this bulk could exist apart from gravitation fields some sterile particles, like heavy right-handed neutrinos which could interact with light left-handed neutrinos. The Majorana neutrino can travel in the bulk?! For this we should introduce a new space time-symmetry which generalizes the usual D-Lorentz symmetry.

The existence of the Majorana fermion matter in nature can give the further development in the understanding of the Lorentz symmetry and matter-antimatter symmetry, the geometrical origin of the gauge symmetries of the Standard Model, 3-quark-lepton family problems, dark matter and dark energy problems in Cosmology. For example, embedding the Majorana neutrino into the higher dimensional space-time we need to find a generalization of relativistic Dirac-Majorana equation which should not contradict to low energy experiments in which the properties of neutrino are known very well! There could be the different ways of embedding the large extra-dimensions cycles according some new symmetries, what can give us new phenomena in neutrino physics, such as a possile new $SO(1,1)$ boost at high energies of neutrino.. The embedding the new symmetries (ternary,...)open the window into the extra-dimensional world with $D > 3 + 1$, gives us renormalizable theories in the space-time with $\text{Dim}=5,6,\dots$ similarly as Poincar/'e symme-try with internal gauge symmetries gave the renormalizability of quantum field theories in $D=4$. The D -dimensional binary Lorentz groups cannot allow to go into the $D > 4$ world, *i.e.* to build the renormalizable theories for the large space-time geometry with dimension $D > 4$.. It seems very plausible that using such ternary symmetries will appear a real possibility to overcome the problems with quantization of a membrane theory and what could be a further progress beyond the string/SS theories. Also these new ternary algebras could be related with some new SUSY approaches. Getting the renormalizable quantum field theories in $D > 4$ space-time we could find the point-like limits of the string and membrane theories for some new dimensions $D > 4$.

Many interesting and important attempts have been done to solve the problems in extensions of the Standard Model in terms of Cartan-Lie algebras, for example: left-right extension, horizontal symmetries, $SU(5)$, $SO(10)$, $E(6)$, $E(8)$, Grand Unified theories, SUSY and SUGRA models.

At last in the superstring/ D -branes approaches it was suggested a way to construct Theory of Everything. The theory of superstrings is also based on the binary Lie groups,

in particular on the D-dimensional Lorentz group, and therefore the description of the Standard Model in the superstrings approach did not bring us to success. In our opinion, one of the main problems with the superstrings approaches is the inadequate external symmetry at the string scale, $M_{\text{str}} \gg M_{\text{SM}}$, the D-Lorentz symmetry must be generalized. This problem exists also for GUTs.

So, all modern theories based on the binary Lie algebras have a common property since the algebras/symmetries are related with some invariant quadratic forms.

In all these approaches there were used a wide class of simple classical Lie algebras, whose Cartan-Killing classification contains four infinite series $A_r = sl(r+1)$, $B_r = so(2r+1)$, $C_r = sp(2r)$, $D_r = so(2r)$ and five exceptional algebras G_2 , F_4 , E_6 , E_7 , E_8 [19, 20, 21, 29]. There were used some ways to study such classification. We can remind some of them, one way is through the theory of numbers and Clifford algebras, the second is the geometrical way, and at last, the third is through the theory of Cartan matrices and Dynkin diagrams (see [27, 21]).

Note, that the theory of simple roots allows to reconstruct all root system and, consequently, all commutation relations in the corresponding CLA.

The finite-dimensional Lie algebra g of a compact simple Lie group G is determined by the following commutation relations

$$[T_a, T_b]_{Z_2} = \mathbf{i} f_{abc} T_c, \quad (1)$$

where the basis of generators $\{T_a\}$ of g is assumed to satisfy the orthonormality condition:

$$Tr(T_a T_b) = y \delta_{ab}. \quad (2)$$

The constant y depends on the representation chosen.

The standard way of choosing a basis for g is to define the maximal set h , $[hh] = 0$, of commuting Hermitian generators, H_i , ($i=1,2,\dots,r$).

$$[H_i, H_j] = 0, \quad 1 \leq i, j \leq r. \quad (3)$$

This set h of H_i forms the Abelian Cartan subalgebra (CSA). The dimension r is called the rank of g (or G). Then we can extend a basis taking complex generators $E_{\vec{\alpha}}$, such that

$$[H_i, E_{\vec{\alpha}}] = \alpha_i E_{\vec{\alpha}}, \quad 1 \leq i \leq r. \quad (4)$$

From these commutation relations one can give the so called Cartan decomposition of algebra g with respect to the subalgebra h :

$$g = h \oplus \sum_{\vec{\alpha} \in \Phi} g_{\vec{\alpha}}, \quad (5)$$

where $g_{\vec{\alpha}}$ is one-dimensional vector space, formed by step generator $E_{\vec{\alpha}}$ corresponding to the real r -dimensional vector $\vec{\alpha}$ which is called a root. Φ is a set of all roots.

For each $\vec{\alpha}$ there is one essential step operator $E_{\vec{\alpha}} \in g_{\vec{\alpha}}$ and for $-\vec{\alpha}$ there exist the step operator $E_{-\vec{\alpha}} \in g_{-\vec{\alpha}}$ and

$$E_{-\vec{\alpha}} = E_{\vec{\alpha}}^*. \quad (6)$$

It is convenient to form a basis for r -dimensional root space Φ . It is well-known that a basis $\vec{\alpha}_1, \dots, \vec{\alpha}_r \in \Pi \subset \Phi$ can be chosen in such a way that for any root $\vec{\alpha} \in \Phi$ one can get that

$$\vec{\alpha} = \sum_{i=1}^{i=r} n_i \vec{\alpha}_i, \quad (7)$$

where each $n_i \in \mathbb{Z}$ and either $n_i \leq 0, 1 \leq i \leq r$, or $n_i \geq 0, 1 \leq i \leq r$. In the former case $\vec{\alpha}$ is said to be positive ($\Phi^+ : \vec{\alpha} \in \Phi^+$) or in the latter case is negative ($\Phi^- : \vec{\alpha} \in \Phi^-$).

Such basis is basis of simple roots.

So if such a basis is constructed one can see that for each $\vec{\alpha} \in \Phi^+ \subset \Phi$, the set of the non-zero roots Φ contains itself $-\vec{\alpha} \in \Phi^- \subset \Phi$, such that

$$\Phi = \Phi^+ \cup \Phi^-, \quad \Phi^- = -\Phi^+. \quad (8)$$

To complete the statement of algebra g we need to consider $[E_{\vec{\alpha}}, E_{\vec{\beta}}]$ for each pair of roots $\vec{\alpha}, \vec{\beta}$. From the Jacobi identity one can get

$$[H_i, [E_{\vec{\alpha}}, E_{\vec{\beta}}]] = (\alpha_i + \beta_i)[E_{\vec{\alpha}}, E_{\vec{\beta}}]. \quad (9)$$

From this one can get

$$\begin{aligned} [E_{\vec{\alpha}}, E_{\vec{\beta}}] &= N_{\vec{\alpha}, \vec{\beta}} E_{\vec{\alpha} + \vec{\beta}}, & \text{if } \vec{\alpha} + \vec{\beta} &\in \Phi \\ &= 2 \frac{\vec{\alpha} \cdot \vec{H}}{\langle \vec{\alpha}, \vec{\alpha} \rangle}, & \text{if } \vec{\alpha} + \vec{\beta} &= 0, \\ &= 0, & \text{otherwise.} \end{aligned} \quad (10)$$

All this choice of generators is called a Cartan-Weyl basis. For each root $\vec{\alpha}$,

$$\left\{ E_{\vec{\alpha}}, \quad 2 \frac{\vec{\alpha} \cdot \vec{H}}{\langle \vec{\alpha}, \vec{\alpha} \rangle}, \quad E_{-\vec{\alpha}} \right\} \quad (11)$$

form an $su(2)$ subalgebra, isomorphic to

$$\{ I_+, \quad 2I_3, \quad I_- \}, \quad (12)$$

where

$$[I_+, I_-] = 2I_3, \quad [I_3, I_{\pm}] = \pm I_{\pm} \quad (13)$$

with

$$I_+^* = I_-, \quad I_3^* = I_3. \quad (14)$$

As consequence one can expect that the eigenvalues of $2 \frac{\vec{\alpha} \cdot \vec{H}}{\langle \vec{\alpha}, \vec{\alpha} \rangle}$ are integral, i.e.:

$$2 \frac{\langle \vec{\alpha}, \vec{\beta} \rangle}{\langle \vec{\alpha}, \vec{\alpha} \rangle} \in \mathbb{Z} \quad (15)$$

for all roots $\vec{\alpha}, \vec{\beta}$.

As the examples one can consider one can consider the root systems for $su(3)$ of rank 2 (see $su(3)$ root system Fig. ??) [21].

Now we introduce the plus-step operators:

$$Q_1 = Q_I^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad Q_2 = Q_{II}^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad Q_3 = Q_{III}^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad (16)$$

and on the minus-step operators:

$$Q_4 = Q_I^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad Q_5 = Q_{II}^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad Q_6 = Q_{III}^- = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (17)$$

We choose the following 3-diagonal operators :

$$H_3 = Q_7 = Q_I^0 = \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\sqrt{\frac{2}{3}} \end{pmatrix} \quad H_8 = Q_8 = Q_{II}^0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad Q_0 = Q_{III}^0 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ 0 \end{pmatrix}$$

For $su(3)$ algebra the positive roots can be chosen as

$$\vec{\alpha}_1 = (1, 0), \quad \vec{\alpha}_2 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad \vec{\alpha}_1 + \vec{\alpha}_2 = \vec{\alpha}_3 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \quad (19)$$

with

$$\langle \vec{\alpha}_1, \vec{\alpha}_1 \rangle = \langle \vec{\alpha}_2, \vec{\alpha}_2 \rangle = 1 \quad \text{and} \quad \langle \vec{\alpha}_1, \vec{\alpha}_2 \rangle = -\frac{1}{2}. \quad (20)$$

$$\begin{aligned} [\vec{H}, Q_{\pm\vec{\alpha}_1}] &= \pm(1, 0) Q_{\pm\vec{\alpha}_1}; & [Q_{\vec{\alpha}_1}, Q_{-\vec{\alpha}_1}] &= 2(1, 0) \cdot \vec{H}; \\ [\vec{H}, Q_{\pm\vec{\alpha}_2}] &= \pm\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) Q_{\pm\vec{\alpha}_2}, & [Q_{\vec{\alpha}_2}, Q_{-\vec{\alpha}_2}] &= 2\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \cdot \vec{H}; \\ [\vec{H}, Q_{\pm\vec{\alpha}_3}] &= \pm\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) Q_{\pm\vec{\alpha}_3}, & [Q_{\vec{\alpha}_3}, Q_{-\vec{\alpha}_3}] &= 2\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \cdot \vec{H}; \end{aligned} \quad (21)$$

where $\vec{H} = (H_3, H_8)$. The commutation relations of step operators can be also easily written:

$$\begin{aligned} [Q_{\vec{\alpha}_1}, Q_{\vec{\alpha}_2}] &= Q_{\vec{\alpha}_3}, & [Q_{\vec{\alpha}_1}, Q_{\vec{\alpha}_3}] &= [Q_{\vec{\alpha}_2}, Q_{\vec{\alpha}_3}] = 0 \\ [Q_{\vec{\alpha}_1}, Q_{-\vec{\alpha}_3}] &= Q_{-\vec{\alpha}_2}, & [Q_{\vec{\alpha}_2}, Q_{-\vec{\alpha}_3}] &= Q_{-\vec{\alpha}_1}. \end{aligned} \quad (22)$$

For A_2 algebra the nonzero roots can be also expressed through the orthonormal basis $\{\vec{e}_i\}, i = 1, 2, 3$, in which all the roots are lying on the plane ortogonal to the vector $\vec{k} = 1 \cdot \vec{e}_1 + 1 \cdot \vec{e}_1 + 1 \cdot \vec{e}_1$, i.e. $\vec{k} \cdot \vec{\alpha} = 0$. Then for this algebra the positive roots are the following:

$$\vec{\alpha}_1 = \vec{e}_1 - \vec{e}_2, \quad \vec{\alpha}_2 = \vec{e}_2 - \vec{e}_3, \quad \vec{\alpha}_3 = \vec{e}_1 - \vec{e}_3. \quad (23)$$

This basis can be practically used in general case to give the complete list of simple finite dimensional Lie algebras

$$\begin{array}{llll} SU(n) & \pm(e_i - e_j) & 1 \leq i \leq j \leq n & 0 \quad (n-1) \\ SO(2n) & \pm e_i \pm e_j & 1 \leq i \leq j \leq n & 0 \quad n \\ SO(2n+1) & \pm e_i \pm e_j & 1 \leq i \leq j \leq n & 0 \quad n \\ & \pm e_i & 1 \leq i \leq n & \\ Sp(n) & \pm e_i \pm e_j & 1 \leq i \leq j \leq n & 0 \quad n \\ & \pm 2e_i & 1 \leq i \leq n & \end{array} \quad (24)$$

Since $su(n)$ is the Lie algebra of traceless $n \times n$ anti-Hermitian matrices there being $(n-1)$ linear independent diagonal matrices. Let $h_{kl} = (e_{kk} - e_{k+1,k+1})$, $k = 1, \dots, n-1$ be the choice of the diagonal matrices, and let e_{pq} for $p, q = 1, \dots, n$, $p < q$ be the remaining the basis elements:

$$(e_{pq})_{ks} = \delta_{kp} \delta_{sq}. \quad (25)$$

It is easily see that the simple finite-dimensional algebra G can be encoded in the $r \times r$ Cartan matrix

$$A_{ij} = \frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}, \quad 1 \leq i, j \leq r, \quad (26)$$

with simple roots, α_i , which generally obeys to the following rules:

$$\begin{array}{ll} \mathbb{A}_{ii} & = 2 \\ \mathbb{A}_{ij} & \leq 0 \\ \mathbb{A}_{ij} = 0 & \mapsto \mathbb{A}_{ji} = 0 \\ \mathbb{A}_{ij} & \in \mathbb{Z} = 0, 1, 2, 3 \\ Det \mathbb{A} & > 0. \end{array} \quad (27)$$

The rank of \mathbb{A} is equal to r .

$$\begin{aligned}
A_r : Det(\mathbb{A}) &= (r+1), \\
D_r : Det(\mathbb{A}) &= 4, \\
B_r : Det(\mathbb{A}) &= 2, \\
C_r : Det(\mathbb{A}) &= 2, \\
F_4 : Det(\mathbb{A}) &= 1, \\
G_2 : Det(\mathbb{A}) &= 1, \\
E_r : Det(\mathbb{A}) &= 9-r, \quad r = 6, 7, 8.
\end{aligned} \tag{28}$$

Also using theory of the simple roots and Cartan matrices the list of simple killing-Cartan-Lie algebras can be encoded in the Dynkin diagram.

The Dynkin diagram of g is the graph with nodes labeled $1 \dots, r$ in a bijective correspondence with the set of the simple roots, such that nodes i, j with $i \neq j$ are joined by n_{ij} lines, where $n_{ij} = \mathbb{A}_{ij}\mathbb{A}_{ji}, i \neq j$.

One can easily get that $A_{ij}A_{ji} = 0, 1, 2, 3..$ Its diagonal elements are equal 2 and its off-diagonal elements are all negative integers or zero. The information in A is coded into Dynkin diagram which is built as follows: it consists of the points for each simple root $\vec{\alpha}_i$ with points $\vec{\alpha}_i$ and $\vec{\alpha}_j$ being joined by $A_{ij}A_{ji}$ lines, with arrow pointing from $\vec{\alpha}_j$ to $\vec{\alpha}_i$ if $\langle \vec{\alpha}_j, \vec{\alpha}_j \rangle > \langle \vec{\alpha}_i, \vec{\alpha}_i \rangle$.

Obviously, that $\hat{\mathbb{A}}_{ij} = \mathbb{A}_{ij}$ for $1 \leq i, j \leq r$, and $\hat{\mathbb{A}}_{00} = 2$. For generalized Cartan matrix there are two unique vectors a and a^\vee with positive integer components (a_0, \dots, a_r) and $(a_0^\vee, \dots, a_r^\vee)$ with their greatest common divisor equal one, such that

$$\sum_{i=0}^r a_i \hat{\mathbb{A}}_{ji} = 0, \quad \sum_{i=0}^r \hat{\mathbb{A}}_{ij} a_j^\vee = 0. \tag{29}$$

The numbers, a_i and a_i^\vee are called Coxeter and dual Coxeter labels. Sums of the Coxeter and dual Coxeter labels are called by Coxeter h and dual Coxeter numbers h^\vee . For symmetric generalized Cartan matrix the both Coxeter labels and numbers coincide. The components a_i , with $i \neq 0$ are just the components of the highest root of Cartan-Lie algebra. The Dynkin diagram for Cartan-Lie algebra can be get from generalized Dynkin diagram of affine algebra by removing one zero node. The generalized Cartan matrices and generalized Dynkin diagrams allow one-to-one to determine affine Kac-Moody algebras .

2 From classification of Calabi-Yau spaces to the Berger graphs and N-ary algebras

We already know that the superstring GUT s did not bring us an expected success for explanation or understanding as mentioned above many problems of the SM. The main progress in superstrings (strings) was related with understanding that we should go to the extra dimensional geometry with $D > 4$. Also the superstrings turned us again to study the geometrical approach, which has brought in XIX century the big progress in physics. This geometrical objects, Calabi-Yau spaces with $SU(3)$ holonomy, appeared in the process of the compactification of the heterotic $E(8) \times E(8)$ 10-dimensional superstring on $M_4 \otimes K_6$ space or study the duality between 5 superstring/M/F theories. Mathematics [15] discovered such objects using the holonomy principle. To get $K_6 = CY_3$, the main constraint on the low energy physics was to conserve a very important property of the internal symmetry, i.e., to build a grand unified theory with $N = 1$ supersymmetry. It has been got the very important result that the infinite series of the compact complex CY_n spaces with $SU(n)$ holonomy can be described by algebraic way of the reflexive numbers (projective weight vectors) . This series starts from the torus with complex dimension $d = 1$ and $K3$ spaces with complex dimension $d = 2$, with $SU(1)$ and $SU(2)$ holonomy groups, respectively. We would like to stress that consideration of the extended string theories leads us to a new geometrical objects, with more interesting properties than the well-known symmetric homogeneous spaces using in the SCM. For example, the $K3 = CY_2$ -singularities are responsible for producing Cartan-Lie ADE-series matter using in the SM. The singularities of CY_n spaces with $n \geq 3$ should be responsible with producing of new algebras and symmetries beyond Cartan-Lie and which can help us to solve the questions of the SM and SCM [22, 13]. This geometrical direction is related with Felix Klein,s old ideas in his Erlangen program which promotes the very closed link between geometrical objects and symmetries.

As we already said that there are some ways to construct ternary algebras and symmetries. One of them is linked to the theory of numbers. The fundamental property of the simple KCLA classification is the Abelian Cartan subalgebra and the circumstance that for each step generator of an algebra you can build the $su(2)$ -subalgebra. For example, it is well known that binary complex numbers of module 1 are related to Abelian $U(1) = S^1$ group. The imaginary quaternion units are related to the $su(2)$ algebra and the unit quaternions are related to the $SU(2) = S^3$ group. And at last octonions are related to the $G(2)$ group. So, our way is to consider ternary algebras and groups based on the ternary generalization of binary numbers (real, complex, quaternions, octonions) (see [?, ?, 13]).

The second geometrical way is very closed related to the symmetries of some geometrical objects. For example, it is well known Cartan-Lie symmetries are closely connected with homogeneous symmetrical spaces. Due to the superstring approach physiciens have got a big interet to the Calabi-Yau geometry. It was shown that the spaces of dimension $n = 2$, K_3 -spacesm are closely related to the Cartan-Lie algebras. Then it was proposed that such spaces of $n = 3, 4, \dots$ could be related to the new n -ary algebras and symmetries. We plan to study this question for $n = 3$ case through the Berger graphs, which can be

found in CY_3 reflexive Newton polyhedra. We determine the Berger graphs based on the AENV-algebraic classification of CY_n spaces. Actually, the Berger graphs are directly determined by reflexive projective weight vectors, which determine the CY -spaces. The Calabi-Yau spaces with $SU(n)$ holonomy can be studied by the algebraic way through the integer lattice where one can construct the Newton reflexive polyhedra or the Berger graphs. Our conjecture is that the Berger graphs can be directly related with the n -ary algebras. To find such algebras we study the n -ary generalization of the well-known binary norm division algebras, \mathbb{R} , \mathbb{C} , \mathbb{H} , \mathbb{O} , which helped to discover the most important "minimal" binary simple Lie groups, $U(1)$, $SU(2)$ and $G(2)$. As the most important example, we consider the case $n = 3$, which gives the ternary generalization of quaternions (octonions), 3^n , $n = 2, 3$, respectively. The ternary generalization of quaternions is directly related to the new ternary algebra (group) which are related to the natural extensions of the binary $su(3)$ algebra ($SU(3)$ group).

Our interest in ternary algebras and symmetries started from the study of the geometry based on the holonomy principle, discovered by Berger [15]. The CY_n spaces with $SU(n)$ holonomy [?, ?] have a special interest for us. Our conjecture [18, 22] is that CY_3 (CY_n) spaces are related to the ternary (n -ary) symmetries, which are natural generalization of the binary Cartan–Killing–Lie symmetries.

The holonomy group H is one of the main characteristic of an affine connection on a manifold M . The definition of holonomy group is directly connected with parallel transport along the piece-smooth path joining two points $x \in M$ and $y \in M$. For a connected n -dimensional manifold M with Riemannian metric g and Levi-Civita connection the parallel transport along using the connection defines the isometry between the scalar products on the tangent spaces $T_x M$ and $T_y M$ at the points x and y . So for any point $x \in M$ one can represent the set of all linear automorphisms of the associated tangent spaces $T_x M$ which are induced by parallel translation along x -based loop.

If a connection is locally symmetric then its holonomy group equals to the local isotropy subgroup of the isometry group G . Hence, the holonomy group classification of these connections is equivalent to the classification of symmetric spaces which was done completely long ago [?] The full list of symmetric spaces is given by the theory of Lie groups through the homogeneous spaces $M = G/H$, where G is a connected group Lie acting transitively on M and H is a closed connected Lie subgroup of G , what determines the holonomy group of M . Symmetric spaces have a transitive group of isometries. The known examples of symmetric spaces are \mathbb{R}^n , spheres S^n , \mathbb{CP}^n etc. There is a very interesting fact that Riemannian spaces (M, g) is locally symmetric if and only if it has constant curvature $\nabla R = 0$.

If we consider irreducible (compact, simply-connected) Riemannian manifolds one can find there classical manifolds, the symmetric spaces, determined by following form G/H , where G is a compact Lie group and H is the holonomy group itself. These spaces are completely classified and their geometry is well-known. But there exists non-symmetric irreducible Riemannian manifolds with the following list of holonomy groups H of M .

Firstly, in 1955, Berger presented the classification of irreducibly acting matrix Lie groups occurred as the holonomy of a torsion free affine connection.

The set of homogeneous polynomials of degree d in the complex projective space

\mathbf{CP}^n defined by the vector \vec{k}_{n+1} with $d = k_1 + \dots + k_{n+1}$ defines a convex polyhedron, whose intersection with the integer lattice corresponds to the exponents of the monomials of the equation. Batyrev found the properties of such polyhedra like reflexivity which directly links these polyhedra to the Calabi-Yau equations. Therefore, instead of studying the complex hypersurfaces directly, firstly, one can study the geometrical properties of such polyhedra..

One of the main results in the Universal Calabi-Yau Algebra (UCYA) is that the reflexive weight vectors (RWVs) \vec{k}_n of dimension n can be obtained directly from lower-dimensional RWVs $\vec{k}_1, \dots, \vec{k}_{n-r+1}$ by algebraic constructions of arity r [16]. One of the important consequences of UCYA one can see the lattice structure connected to the Berger graphs. In K3 case it was shown that the Newton reflexive polyhedra are constructed by pair of plane Berger graphs coinciding to the Dynkin diagrams of CLA algebra. In CY_3 the four dimensional reflexive polyhedra are constructed from triple of Berger graphs which by our opinion could be related to the new algebra, which can be the ternary generalizations of binary CLAs:

$$\begin{aligned}
\vec{k}_1 &= (0, \dots, 1)[1], & \rightarrow & A_r^{(1)}(K3), & TA_r^{(1)}(CY_3), \dots \\
\vec{k}_2 &= (0, \dots, 1, 1)[2], & \rightarrow & D_r^{(1)}(K3), & TD_r^{(1)}(CY_3), \dots \\
\vec{k}_3 &= (0, \dots, 1, 1, 1)[3], & \rightarrow & E_6^{(1)}(K3), & TE_6^{(1)}(CY_3), \dots \\
\vec{k}_3 &= (0, \dots, 1, 1, 2)[4], & \rightarrow & E_7^{(1)}(K3), & TE_7^{(1)}(CY_3), \dots \\
\vec{k}_3 &= (0, \dots, 1, 2, 3)[6] & \rightarrow & E_8^{(1)}(K3), & TE_8^{(1)}(CY_3), \dots
\end{aligned} \tag{30}$$

So, the other important success of UCYA is that it is naturally connected to the invariant topological numbers, and therefore it gives correctly all the double-, triple-, and etc. intersections, and, correspondingly, all graphs, which are connected with affine algebras.

It was shown in the toric-geometry approach how the Dynkin diagrams of affine Cartan-Lie algebras appear in reflexive K3 polyhedra [24]. Moreover, it was found in [16], using examples of the lattice structure of reflexive polyhedra for CY_n : $n \geq 2$ with elliptic fibres that there is an interesting correspondence between the five basic RWVs (30) and Dynkin diagrams for the five ADE types of Lie algebras: A, D and $E_{6,7,8}$. For example, these RWVs are constituents of composite RWVs for K3 spaces, and the corresponding K3 polyhedra can be directly constructed out of certain Dynkin diagrams, as illustrated in . In each case, a pair of extended RWVs have an intersection which is a reflexive plane polyhedron, and one vector from each pair gives the left or right part of the three-dimensional reflexive polyhedron, as discussed in detail in [16].

One can illustrate this correspondence on the example of RWVs, $\vec{k}_3 = (k_1, k_2, k_3)[d_{\vec{k}}] = (111)[3], (112)[4], (123)[6]$, for which we show how to build the $E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$ Dynkin diagrams, respectively. Let take the vector $\vec{k}_3 = (111)[3]$. To construct the Dynkin diagram one should start from one common node, V^0 , which will give start to $n=3$ (= dimension of the vector) line-segments. To get the number of the points-nodes p on each line one should divide $d_{\vec{k}}$ on k_i , $i = 1, 2, 3$, so $p_i = d_{\vec{k}}/k_i$ (here we considere the cases

when all divisions are integers). One should take into account, that all lines have one common node V^0 . The numbers of the points equal to $n \cdot (d_{\vec{k}}/k_i - 1) + 1$. Thus, one can check, that for all these three cases there appear the $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$ graphs, respectively. Moreover, one can easily see how to reproduce for all these graphs the Coxeter labels and the Coxeter number. Firstly, one should prescribe the Coxeter label to the common point V^0 . It equals to $\max_i\{p_i\}$. So in our three cases the maximal Coxeter label, prescribing to the common point V^0 , is equal 3, 4, 6, respectively. Starting from the Coxeter label of the node V^0 , one can easily find the Coxeter numbers of the rest points in each line. Note that this rule will help us in the cases of higher dimensional CY_d with $d \geq 3$, for which one can easily represent the corresponding polyhedron and graphs without computers.

Similarly, the huge set of five-dimensional RWVs \vec{k}_5 in 4242 CY_3 chains of arity 2 can be constructed out of the five RWVs already mentioned plus the 95 four-dimensional K3 RWVs \vec{k}_4 . In this case, reflexive 4-dimensional polyhedra are also separated into three parts: a reflexive 3-dimensional intersection polyhedron and ‘left’ and ‘right’ graphs. By construction, the corresponding CY_3 spaces are seen to possess K3 fibre bundles.

We illustrate the case of one such arity-2 K3 example [16, 17]. In this case, a reflexive K3 polyhedron is determined by the two RWVs $\vec{k}_1 = (1)[1]$ and $\vec{k}_3 = (1, 2, 3)[6]$. As one can see, this K3 space has an elliptic Weierstrass fibre, and its polyhedron, determined by the RWV $\vec{k}_4 = (1, 0, 0, 0) + (0, 1, 2, 3) = (1, 1, 2, 3)[7]$, can be constructed from two diagrams, $A_6^{(1)}$ and $E_8^{(1)}$, depicted to the left and right of the triangular Weierstrass skeleton. The analogous arity-2 structures of all 13 eldest K3 RWVs [16].

The extra uncompactified dimensions make quantum field theories with Lorentz symmetry much less comfortable, since the power counting is worse. A possible way out is to suppose that the propagator is more convergent than $1/p^2$, such a behaviour can be obtained if we consider, instead of binary symmetry algebra, algebras with higher order relations (*That is, instead of binary operations such as addition or product of 2 elements, we start with composition laws that involve at least n elements of the considered algebra, n -ary algebras*). For instance, a ternary symmetry could be related with membrane dynamics. To solve the Standard Model problems we suggested to generalize their external and internal binary symmetries by addition of ternary symmetries based on the ternary algebras [18, 22]. For example, ternary symmetries seem to give very good possibilities to overcome the above-mentioned problems, *i.e.* to make the next progress in understanding of the space-time geometry of our Universe. We suppose that the new symmetries beyond the well-known binary Lie algebras/superalgebras could allow us to build the renormalizable theories for space-time geometry with dimension $D > 4$. It seems very plausible that using such ternary symmetries will offer a real possibility to overcome the problems of quantization of membranes and could be a further progress beyond string theories.

Our interest in the new n -ary algebras and their classification started from a study of infinite series of CY_n spaces characterized by holonomy groups [15]. More exactly, the CY_n space can be defined as the quadruple (M, J, g, Ω) , where (M, J) is a complex compact n -dimensional manifold.

A CY_{n-2} space can be realized as an algebraic variety \mathcal{M} in a weighted projective space $\mathbb{CP}^{n-1}(\vec{k})$ where the weight vector reads $\vec{k} = (k_1, \dots, k_n)$.

The points in \mathbb{CP}^{n-1} satisfy the property of projective invariance $\{x_1, \dots, x_n\} \approx \{\lambda^{k_1} x_1, \dots, \lambda^{k_n} x_n\}$ leading to the constraint $\vec{m} \cdot \vec{k} = d_k$.

The classification of CY_n can be done through the reflexivity of the weight vectors \vec{k} (reflexive numbers), which can be defined in terms of the Newton reflexive polyhedra [24] or Berger graphs [18]. The Newton reflexive polyhedra are determined by the exponents of the monomials participating in the CY_n equation [24]. The term "reflexive" is related with the mirror duality of Calabi–Yau spaces and the corresponding Newton polyhedra [24]. The Berger graphs can be constructed directly through the reflexive weight numbers $\vec{k} = (k_1, \dots, k_{n+2})[d_k]$ by the procedure shown in [18, 22]. According to the universal algebraic approach [16] one can find a section in the reflexive polyhedron and, according to the n -arity of this algebraic approach, the reflexive polyhedron can be constructed from 2-, 3-, ... Berger graphs. It was conjectured that the Berger graphs might correspond to n -ary Lie algebras [18, 22]. In these articles we tried to decode those Berger graphs by using the method of the "simple roots".

All modern theories based on the binary Lie algebras have the common property since the algebras/symmetries are related with some invariant quadratic forms. Ternary algebras/symmetries should be linked also with certain cubic invariant forms. Our interest to the new n -ary algebras and their classification started from study of infinite series of CY_n spaces characterized by holonomy groups [15]. More exactly, the CY_n space can be defined as the quadruple (M, J, g, Ω) , where (M, J) is a complex compact n -dimensional manifold with complex structure J , g is a kahler metrics with $SU(n)$ holonomy group holonomy, and $\Omega_n = (n, 0)$ and $\bar{\Omega}_n = (0, n)$ are non-zero parallel tensors which called by the holomorphic volume forms.

A CY space can be realized as an algebraic variety \mathcal{M} in a weighted projective space $\mathbb{CP}^{n-1}(\vec{k})$ where the weight vector reads $\vec{k} = (k_1, \dots, k_n)$. This variety is defined by

$$\mathcal{M} \equiv (\{x_1, \dots, x_n\} \in \mathbb{CP}^{n-1}(\vec{k}) : \mathcal{P}(x_1, \dots, x_n) \equiv \sum_{\vec{m}} c_{\vec{m}} x^{\vec{m}} = 0), \quad (31)$$

i.e., as the zero locus of a quasi-homogeneous polynomial of degree $d_k = \sum_{i=1}^n k_i$, with the monomials being $x^{\vec{m}} \equiv x_1^{m_1} \dots x_n^{m_n}$. The points in \mathbb{CP}^{n-1} satisfy the property of projective invariance $\{x_1, \dots, x_n\} \approx \{\lambda^{k_1} x_1, \dots, \lambda^{k_n} x_n\}$ leading to the constraint $\vec{m} \cdot \vec{k} = d_k$.

For classifying and decoding the new graphs one can use the following rules:

1. to classify the graphs one can do according to the arity, *i.e.*
for arity 2 here can be two graphs, and the points on the left (right) graph should be on the edges lying on one side with respect to the arity 2 intersection
for arity 3 there can be three graphs, which points can be defined with respect to the arity 3 intersections and etc.
for arity r there can be r graphs

2. The graphs should correspond to extension of affine graphs of Kac-Moody algebra

3. The graphs can correspond to an universal algebra with some arities

The first proposal was already discussed before. The second proposal is important because a possible new algebra could be connected very closely with geometry. Loop algebra is a Lie algebra associated to a group of mapping from manifold to a Lie group. Concretely to get affine Kac-Moody it was considered the case where the manifold is the unit circle and group is a matrix Lie group. Here it can be a further geometrical way to generalize the affine Kac-Moody algebra. We will take this in mind, but we will always suppose that the affine property of the new graphs should remain as it was in affine Kac-Moody algebra classification. The affine property means that the matrices corresponding to these algebras should have the determinant equal to zero, and all principal minors of these matrices should be positive definite. The matrices will be constructed with almost the same rules as the generalized Cartan matrices in affine Kac-Moody case. We just make one changing on the some diagonal elements, which can take the value not only 2, but also 3 for CY_3 case (4 for CY_4 case and etc). The third proposal is connected with taking in mind that a new algebra could be an universal algebra, *i.e.* it contains apart from binary operation also ternary,... operations. The suggestion of using a ternary algebra interrelates with the topological structure of CP^2 . This can be used for resolution of CY_3 singularities. It seems that taking into consideration the different dimensions, one can understand very deeply how to extend the notion of

Lie algebras and to construct the so called universal algebras. These algebras could play the main role in understanding of *non-symmetric* Calabi-Yau geometry and can give a further progress in the understanding of high energy physics in the Standard model and beyond.

Our plan is following, at first we study the graphs connected with five reflexive weight vectors, $(1), (11), (111), (112), (123)$ and then, we consider the examples with $K3$ - reflexive weight vectors.

To study the lattice structure of the graphs in reflexive polyhedra one should recall a little bit about Cartan matrices and Dynkin diagrams..

Our reflexive polyhedra allow us to consider new graphs, which we will call Berger graphs, and for corresponding Berger matrices we suggest the following rules:

$$\begin{aligned}
\mathbb{B}_{ii} &= 2 \quad \text{or} \quad 3, \\
\mathbb{B}_{ij} &\leq 0, \\
\mathbb{B}_{ij} = 0 &\mapsto \mathbb{B}_{ji} = 0, \\
\mathbb{B}_{ij} &\in \mathbb{Z}, \\
\text{Det} \mathbb{B} &= 0, \\
\text{Det} \mathbb{B}_{\{(i)\}} &> 0.
\end{aligned} \tag{32}$$

We call the last two restriction the *affine condition*. In these new rules comparing with the generalized affine Cartan matrices we relaxed the restriction on the diagonal element

\mathbb{B}_{ii} , *i.e.* to satisfy the affine conditions we allow also to be

$$\mathbb{B}_{ii} = 3 \text{ for } CY_3, \quad \mathbb{B}_{ii} = 4 \text{ for } CY_4, \quad \text{and etc.} \quad (33)$$

Apart from these rules we will check the coincidence of the graph's labels, which we indicate on all figures with analog of Coxeter labels, what one can get from getting eigenvalues of the Berger matrix.

Let consider the reflexive polyhedron, which corresponds to the K3-fibre CY_3 space and which is defined by two extended vectors, $\vec{k}_L^{ext} = (0, 0, 0, 0, \vec{k}_1), (0, 0, 0, \vec{k}_2), (0, 0, \vec{k}_3)$ and $\vec{k}_R^{ext} = (0, \vec{k}_4)$. The first extended vectors correspond to the RWVs of dimension 1,2 and 3. The second extended vectors correspond to the one of the 95 K3 RWVs. This CY_3 should have the K3 fibre structure.

An interesting subclass of the reflexive numbers is the so-called “simply-laced” numbers (Egyptian numbers). A simply-laced number $\vec{k} = (k_1, \dots, k_n)$ with degree $d = \sum_{i=1}^n k_i$ is defined such that

$$\frac{d}{k_i} \in \mathbb{Z}^+ \text{ and } d > k_i. \quad (34)$$

For these numbers there is a simple way of constructing the corresponding affine Berger graphs together with their Coxeter labels[18, 22]. The Cartan and Berger matrices of these graphs are symmetric. In the well known Cartan case they correspond to the ADE series of simply-laced algebras. In dimensions $n = 1, 2, 3$ the Egyptian numbers are $(1), (1, 1), (1, 1, 1), (1, 1, 2), (1, 2, 3)$. For $n = 4$ among all 95 reflexive numbers 14 are simply-laced Egyptian numbers (see Table).

Let compare the binary affine Dynkin diagrams for \mathcal{E}_6 and affine Berger graph defined by reflexive vector $(0, 1, 1, 1, 1)$.

$\vec{k}_{3,4}^{\text{ext}}$	Rank	h	Casimir(B_{ii})	Determinant
$(0, 1, 1, 1)[3]$	$6(E_6)$	12	6	3
$(0, 1, 1, 2)[4]$	$7(E_7)$	18	8	2
$(0, 1, 2, 3)[6]$	$8(E_8)$	30	12	1
$(0, 0, 1, 1, 1)[3]$	$2_3 + 10 + l$	$18 + 3(l + 1)$	9	3^4
$(0, 0, 1, 1, 2)[4]$	$2_3 + 13 + l$	$32 + 4(l + 1)$	12	4^3
$(0, 0, 1, 2, 3)[6]$	$2_3 + 15l$	$60 + 6(l - 1)$	18	6^2
$(0, 1, 1, 1, 1)[4]$	$1_3 + 11$	28	12	16
$(0, 2, 3, 3, 4)[12]$	$1_3 + 12$	90	36	8
$(0, 1, 1, 2, 2)[6]$	$1_3 + 13$	48	18	9
$(0, 1, 1, 1, 3)[6]$	$1_3 + 15$	54	18	12
$(0, 1, 1, 2, 4)[8]$	$1_3 + 17$	80	24	8
$(0, 1, 2, 2, 5)[10]$	$1_3 + 17$	100	30	5
$(0, 1, 3, 4, 4)[12]$	$1_3 + 17$	120	36	3
$(0, 1, 2, 3, 6)[12]$	$1_3 + 19$	132	36	6
$(0, 1, 4, 5, 10)[20]$	$1_3 + 26$	290	60	2
$(0, 1, 1, 4, 6)[12]$	$1_3 + 24$	162	36	6
$(0, 1, 2, 6, 9)[18]$	$1_3 + 27$	270	54	3
$(0, 1, 3, 8, 12)[24]$	$1_3 + 32$	420	72	2
$(0, 2, 3, 10, 15)[30]$	$1_3 + 25$	420	90	4
$(0, 1, 6, 14, 21)[42]$	$1_3 + 49$	1092	126	1

Table 1: Rank, Coxeter number h , Casimir depending on B_{ii} and determinants for the non-affine exceptional Berger graphs. The maximal Coxeter labels coincide with the degree of the corresponding reflexive simply-laced vector. The determinants in the last column for the infinite series $(0,0,1,1,1)[3]$, $(0,0,1,1,2)[4]$ and $(0,0,1,2,3)[6]$ are independent from the number l of internal binary $B_{ii} = 2$ nodes. The numbers 1_3 and 2_3 denote the number of nodes with $B_{ii} = 3$.

$$\begin{aligned}
\alpha_1 &= e_1 - e_2 \\
\alpha_2 &= e_2 - e_3 \\
\alpha_3 &= e_3 - e_4 \\
\alpha_4 &= e_4 - e_5 - e_9 \\
\alpha_5 &= e_5 - e_6 \\
\alpha_6 &= e_6 - e_7 \\
\alpha_7 &= e_7 - e_8 \\
\alpha_8 &= e_9 - e_{10} \\
\alpha_9 &= -\frac{1}{2}(e_9 - e_{10} + e_1 + e_2 + e_3 + e_4 + e_{11} - e_{12}) \\
\alpha_{10} &= e_{11} - e_{12} \\
\alpha_{11} &= e_9 + e_{10} \\
\alpha_{12} &= -\frac{1}{2}(e_9 + e_{10} - e_5 - e_6 - e_7 - e_8 + e_{11} + e_{12}) \\
\alpha_{13} &= e_{11} + e_{12} = -\alpha_0
\end{aligned} \tag{35}$$

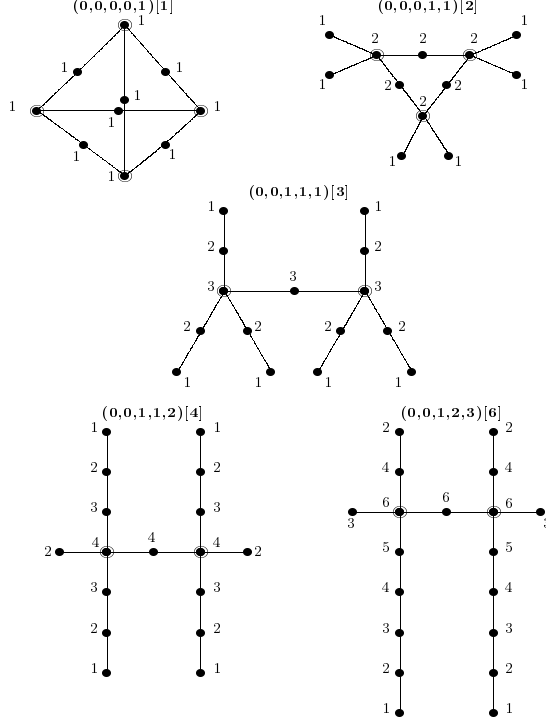


Figure 1: 5 Berger graphs -ternary extensions of ADE Dynkin diagrams

where

$$\begin{aligned}
& 4\alpha_4 + 3(\alpha_3 + \alpha_5 + \alpha_8 + \alpha_{11}) + 2(\alpha_2 + \alpha_6 + \alpha_9 + \alpha_{12}) \\
& + (\alpha_1 + \alpha_7 + \alpha_{10} + \alpha_0) = 0
\end{aligned} \tag{36}$$

$$\left(\begin{array}{ccc|ccc|ccc|ccc}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\
\hline
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2
\end{array} \right)$$

Table 2: *The simple roots of the of CLA \mathcal{E}_6 and Berger algebra defined by reflexive number $k = (0, 1, 1, 1, 1)[4]$.*

\mathcal{E}_6	
\mathcal{BER}	

Note, that the determinant is equal $Det = 4^2$. In general case for CY_d , $d + 2 = n$, which corresponds to the RWV $\vec{k}_n = (1, \dots, 1)[n]$, the determinant of the corresponding non-affine matrices is equal n^{n-2} ($n \geq 3$).

3 \mathbb{C}_N -division numbers and N-ary algebras

We want to find an example of ternary non-Abelian algebra and to understand the mechanism of appearing in Cartan matrix $B_{ii} = 3$. For this we will go to the study of ternary division algebras. Historically the discovery of Killing-Cartan-Lie algebras was closely related to the four norm division \mathbb{C}_2 algebras, $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, i.e. real numbers, complex numbers, quaternions and octonions, respectively [4, 5, 19, 6].

An algebra A will be a vector space that is equipped with a bilinear map $m : A \times A \rightarrow A$ called by multiplication and a nonzero element $1 \in A$ called the unit such that $m(1, a) = m(a, 1) = a$. A normed division algebra is an algebra A that is also a normed vector space with $|ab| = |a||b|$. So \mathbb{R}, \mathbb{C} are the commutative associative normed algebras, \mathbb{H} is noncommutative associative normed algebra. \mathbb{O} are the octonions- a non-associative alternative algebra. An algebra is alternative if $a(ab) = a^2b$ and $(ab)b = ab^2$ ($a(ba) = (ab)a$). An alternative division algebra has unity and inverse element. The only alternative division binary algebras over \mathbb{R} are $\mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{O}$.

An algebra A will be a vector space that is equipped with a bilinear map $f : A \times A \rightarrow A$ called by multiplication and a nonzero element $1 \in A$ called the unit, such that $f(1, a) = f(a, 1) = a$. These algebras admit an anti-involution (or conjugation) $(a^*)^* = a$ and $(ab)^* = b^*a^*$. A norm division algebra is an algebra A that is also a normed vector space with $N(ab) = N(a)N(b)$. Such algebras exist only for $n = 1, 2, 4, 8$ dimensions where the following identities can be obtained:

$$(x_1^2 + \dots + x_n^2)(y_1^2 + \dots + y_n^2) = (z_1^2 + \dots + z_n^2) \quad (37)$$

The doubling process, which is known as the Cayley-Dickson process, forms the sequence of division algebras

$$\mathbb{R} \rightarrow \mathbb{C} \rightarrow \mathbb{H} \rightarrow \mathbb{O}. \quad (38)$$

Note that next algebra is not a division algebra. So $n = 1$ \mathbb{R} and $n = 2$ \mathbb{C} these algebras are the commutative associative normed division algebras. The quaternions, \mathbb{H} , $n = 4$ form the non-commutative and associative norm division algebra. The octonion algebra $n = 8$, \mathbb{O} is a non-associative alternative algebra. If the discovery of complex numbers took a long period about some centuries years, the discovery of quaternions and octonions was made in a short time, in the middle of the XIX century by W. Hamilton [4], and by J. Graves and A. Cayley [5]. The complex numbers, quaternions and octonions can be presented in the general form:

$$\hat{q} = x_0 e_0 + x_p e_p, \quad \{x_0, x_p\} \in \mathbb{R}, \quad (39)$$

where $p = 1$ and $e_1 \equiv \mathbf{i}$ for complex numbers \mathbb{C} , $p = 1, 2, 3$ for quaternions \mathbb{H} , and $p = 1, 2, \dots, 7$ for \mathbb{O} . The e_0 is as unit and all e_p are imaginary units with conjugation $\bar{e}_p = -e_p$. For quaternions we have the main relation

$$e_m e_p = -\delta_{mp} + f_{mpt} e_t, \quad (40)$$

where δ_{mp} and $f_{mpl} \equiv \epsilon_{mpl}$ are the well-known Kronecker and Levi-Cevita tensors, respectively. For octonions the completely antisymmetric tensor $f_{mpl} = 1$ for the following seven triple associate cycles:

$$\{mpl\} = \{123\}, \{145\}, \{176\}, \{246\}, \{257\}, \{347\}, \{365\}. \quad (41)$$

There are also 28 non-associate cycles. Each triple accociate cycle corresponds to a quaternionic subalgebra. These algebras have a very close link with geometry. For example, the unit elements $x^2 = 1, x \in \mathbb{R}, |\hat{q}| = x_0^2 + x_1^2 = 1$ in \mathbb{C}_1 , $|\hat{q}| = x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$ in \mathbb{H}_1 , $|\hat{q}| = x_0^2 + x_1^2 + \dots + x_7^2 = 1$ in \mathbb{O}_1 , define the spheres, S^0, S^1, S^3, S^7 , respectively.

If the binary alternative division algebras (real numbers, complex numbers, quaternions, octonions) over the real numbers have the dimensions $2^n, n = 0, 1, 2, 3, 4, \dots$, the ternary algebras have the following dimensions $3^n, n = 0, 1, 2, 3, 4, \dots$, respectively:

$$\begin{array}{lcl} \mathbb{R} : & 2^0 = 1 & \mathbb{R} : & 3^0 = 1 \\ \mathbb{C} : & 2^1 = 1 + 1 & \mathbb{TC} : & 3^1 = 1 + 1 + 1 \\ \mathbb{Q} : & 2^2 = 1 + 2 + 1 & \mathbb{TQ} : & 3^2 = 1 + 2 + 3 + 2 + 1 \\ \mathbb{O} : & 2^3 = 1 + 3 + 3 + 1 & \mathbb{TO} : & 3^3 = 1 + 3 + 6 + 7 + 6 + 3 + 1 \\ \mathbb{S} : & 2^4 = 1 + 4 + 6 + 4 + 1 & \mathbb{TS} : & 3^4 = 1 + 4 + 10 + 16 + 19 + 16 + 10 + 4 + 1 \end{array} \quad (42)$$

In the last line one can see the sedenions which are do not produce division algebra. For both cases we have the unit element e_0 and the n basis elements:

$$\mathbb{R} \rightarrow \mathbb{TC} \rightarrow \mathbb{TQ} \rightarrow \mathbb{TO} \rightarrow \mathbb{TS} \rightarrow \dots \quad (43)$$

The complex numbers is 2-dimensional algebra with basis e_0 and $e_1 \equiv \mathbf{i}$,

$$\mathbb{C} = \mathbb{R} \oplus \mathbb{R}e_1, \quad (44)$$

where $e_0^2 = e_0, e_1e_0 = e_0e_1 = e_1$ and e_1 is the imaginary unit, $\mathbf{i}^2 = -e_0$. Considering one additional basis imaginary unit element $e_2 \equiv \mathbf{j}$ in the Dickson-Cayley doubling process one can get the quaternions,

$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j. \quad (45)$$

It means that quaternions can be considered as a pair of complex numbers:

$$q = (a + \mathbf{i}b) + j(c + \mathbf{i}d), \quad (46)$$

where

$$j(c + \mathbf{i}d) = (c + \bar{\mathbf{i}}d)j = (c - \mathbf{i}d)j. \quad (47)$$

so, one can see that $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$.

The quaternions

$$q = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3, \quad q \in \mathbb{H}, \quad (48)$$

produce over \mathbb{R} a 4-dimensional norm division algebra where appears the fourth imaginary unit $e_3 = e_1 e_2 \equiv \mathbf{k}$. The main multiplication rules of all these 4-th elements are the following:

$$\begin{aligned} \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 &= -1 \\ \mathbf{ij} = \mathbf{k} \quad \mathbf{ji} &= -\mathbf{k}, \end{aligned} \tag{49}$$

All other identities can be obtained from cyclic permutations of $\mathbf{i}, \mathbf{j}, \mathbf{k}$. The imaginary quaternions $\mathbf{i}, \mathbf{j}, \mathbf{k}$ produce the $su(2)$ algebra. There is the matrix realization of quaternions through the Pauli matrices:

$$\sigma_0, \mathbf{i}\sigma_1, \mathbf{i}\sigma_2, \mathbf{i}\sigma_3, \tag{50}$$

The unit quaternions $q = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in H_1$, $q\bar{q} = 1$, produce the $SU(2)$ group:

$$q\bar{q} = a^2 + b^2 + c^2 + d^2 = 1, \{a, b, c, d\} \in S^3, S^3 \approx SU(2). \tag{51}$$

Similarly, continuing the Cayley-Dickson doubling process $\mathbb{O} = \mathbb{H} \otimes \mathbb{H}$,

$$(x_1, x_2)(y_1, y_2) = (x_1 y_1 - \bar{y}_2 x_2, x_2 \bar{y}_1 + y_2 x_1), \quad ((x, y) = (\bar{x}, \bar{y})), \tag{52}$$

one can build the octonions:

$$\mathbb{O} = \mathbb{Q} \oplus \mathbb{Q}\mathbf{l}, \tag{53}$$

where we introduced new basis element $\mathbf{l} \equiv e_4$.

As result of this process the basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ of \mathbb{H} is complemented to a basis $\{\mathbf{1} = e_0, \mathbf{i} = e_1, \mathbf{j} = e_2, \mathbf{k} = e_3 = e_1 e_2, \mathbf{l} = e_4, \mathbf{il} = e_5 = e_1 e_4, \mathbf{jl} = e_6 = e_2 e_4, \mathbf{kl} = e_7 = e_3 e_4\}$ of \mathbb{O} .

$$o = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7 \tag{54}$$

where we can see the following seven associative cycle triples:

$$\begin{aligned} \{123 : e_1 e_2 = e_3\}, \{145 : e_1 e_4 = e_5\}, \{176 : e_1 e_7 = e_6\}, \{246 : e_2 e_4 = e_6\}, \\ \{257 : e_2 e_5 = e_7\}, \{347 : e_3 e_4 = e_7\}, \{365 : e_3 e_6 = e_5\}. \end{aligned} \tag{55}$$

In order to find new number algebras one can use the method of the classification of the finite groups which is known in literature [?, ?]. On this way one can discover the geometrical objects invariant on the new symmetries, First of all, it will be useful to consider the abelian cyclic groups, $\mathbb{C}_N = \{q^N = 1 | 1, q, q^2, \dots, q^{(N-1)}\}$ of order $N > 2$ *i.e.* $N = 3, 4, 5, \dots$. Following to the the complex numbers where the base unit imaginary element $i^2 = -1$ we will consider two cases: $q^N = \pm 1$. A representation of the group G is a homomorphism of this group into the multiplicative group $GL_m(\Lambda)$ of nonsingular matrices over the field Λ , where $\Lambda = \mathbb{R}, \mathbb{C}$ or etc. The degree of representation is defined by the size of the ring of matrices. If degree is equal one the representation is linear.

For abelian cyclic group \mathbb{C}_N one can easily find the character table, which is $N \times N$ square matrix whose rows correspond to the different characters for a particular conjugacy class, q^α , $\alpha = 0, 1, \dots, N-1$. For cyclic groups \mathbb{C}_N the N irreducible representations are one dimensional (see Table):

$$\left(\begin{array}{c|cccccc} - & 1 & q & \dots & q^\alpha & \dots & q^{N-1} \\ \hline \xi^{(1)} & 1 & 1 & \dots & 1 & \dots & 1 \\ \xi^{(2)} & 1 & \xi_2^{(2)} & \dots & \xi_\alpha^{(2)} & \dots & \xi_N^{(2)} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \xi^{(k)} & 1 & \xi_2^{(k)} & \dots & \xi_\alpha^{(k)} & \dots & \xi_N^{(k)} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \xi^{(N)} & 1 & \xi_2^{(N)} & \dots & \xi_\alpha^{(N)} & \dots & \xi_N^{(N)} \end{array} \right) \quad (56)$$

where the characters can be defined through N -th root of unity. For example, if the character table for \mathbb{C}_N can be summarised as

$$\xi_k^\alpha = \xi^\alpha \exp\{(2\pi i(k-1)(\alpha-1))/N\}, \quad (k, \alpha = 1, 1, 2, \dots, N). \quad (57)$$

Let us consider some examples.

We remind that for the cyclic group \mathbb{C}_2 there are two conjugation classes, 1 and i and two one-dimensional irreducible representations:

$$\begin{array}{c|cc|c} \mathbb{C}_2 & 1 & i & \\ \hline R^{(1)} & 1 & 1 & z \\ R^{(2)} & 1 & -1 & \bar{z} \end{array} .$$

The cyclic group \mathbb{C}_3 has three conjugation classes, q_0 , q and q^2 , and, respectively, three one dimensional irreducible representations, $R^{(i)}$, $i = 1, 2, 3$. We write down the table of their characters, $\xi_l^{(i)}$:

$$\left(\begin{array}{c|ccc} - & 1 & q & q^2 \\ \hline \xi^{(1)} & 1 & 1 & 1 \\ \xi^{(2)} & 1 & j & j^2 \\ \xi^{(3)} & 1 & j^2 & j \end{array} \right) \quad (58)$$

for \mathbb{C}_3 ($j_3 \equiv j = \exp\{2\pi/3\}$).

The cyclic group \mathbb{C}_4 has four conjugation classes, q_0 , q , q^2 and q^3 , and, respectively, four one dimensional irreducible representations, $R^{(i)}$, $i = 1, 2, 3, 4$. We write down the table of their characters, $\xi_l^{(i)}$:

$$\left(\begin{array}{c|cccc} - & 1 & q & q^2 & q^3 \\ \hline \xi^{(1)} & 1 & 1 & 1 & 1 \\ \xi^{(2)} & 1 & i & -1 & -i \\ \xi^{(3)} & 1 & -1 & 1 & -i \\ \xi^{(4)} & 1 & -i & -1 & i \end{array} \right) \quad (59)$$

for \mathbb{C}_4 ($j_4 = \exp\{\pi/2\}$),

Correspondingly, the cyclic group C_6 has six conjugation classes, q_0, q, \dots, q^5 , and, respectively, six one dimensional irreducible representations, $R^{(i)}$, $i = 1, 2, 3, \dots, 6$. We write down the table of their characters, $\xi_l^{(i)}$:

$$\left(\begin{array}{c|cccccc} - & 1 & q & q^2 & q^3 & q^4 & q^5 \\ \hline \xi^{(1)} & 1 & 1 & 1 & 1 & 1 & 1 \\ \xi^{(2)} & 1 & j_6 & j_6^2 & j_6^3 & j_6^4 & j_6^5 \\ \xi^{(3)} & 1 & j_6^2 & j_6^4 & 1 & j_6^2 & j_6^4 \\ \xi^{(4)} & 1 & j_6^3 & 1 & j_6^3 & 1 & j_6^3 \\ \xi^{(5)} & 1 & j_6^4 & j_6^2 & 1 & j_6^4 & j_6^2 \\ \xi^{(6)} & 1 & j_6^5 & j_6^4 & j_6^3 & j_6^2 & j_6 \end{array} \right) \quad (60)$$

and for \mathbb{C}_6 , ($j_6 = \exp\{\pi i/3\}$), respectively.

For all exasmples one can see the orthogonality relations:

$$\langle \xi^{(k)}, \xi^{(l)} \rangle = \delta_{kl}. \quad (61)$$

To check this for the \mathbb{C}_6 case one should take into account the next identities:

$$\begin{aligned} 1 + j_6 + j_6^2 + j_6^3 + j_6^4 + j_6^5 &= 0 \\ j_6 + j_6^3 + j_6^5 &= 0, \quad j_6 - j_6^2 = 1, \\ 1 + j_6^2 + j_6^4 &= 0, \quad j_6^5 - j_6^4 = 1, \end{aligned} \quad (62)$$

or

$$\begin{aligned} j_6 &= \frac{1}{2} + i\frac{\sqrt{3}}{2}, & j_6^2 &= \frac{-1}{2} + i\frac{\sqrt{3}}{2}, & j_6^3 &= -1, \\ j_6^4 &= \frac{-1}{2} - i\frac{\sqrt{3}}{2}, & j_6^5 &= \frac{1}{2} - i\frac{\sqrt{3}}{2}, & j_6^6 &= 1. \end{aligned} \quad (63)$$

We confined ourselves by the case C_6 cyclic group since we supposed to solve the neutrino problem using the consideration of the \mathbb{R}^6 space.

So, the main idea is to use the cyclic groups \mathbb{C}^n and new N-ary algebras/symmetries to find the new geometrical "irreducible" substructures in \mathbb{R}^n spaces, which are not the consequences of the simple extensions of the known structures of Euclidean \mathbb{R}^2 space.

For the ternary complexification of the vector space, R^3 , one uses its cyclic symmetry subgroup $C_3 = R_3$ [?, ?]. In the physical context the elements of the group C_3 are actually spatial rotations through a restricted set of angles, $0, 2\pi/3, 4\pi/3$ around, for example, the x_0 -axis. After such rotations the coordinates, x_0, x_1, x_2 , of the point in R^3 are linearly related with the new coordinates, x'_0, x'_1, x'_2 which can be realized by the 3×3 matrices

corresponding to the C_3 -group transformations. The vector representation D^V is defined through the following three orthogonal matrices:

$$\begin{aligned} R^V(q_0) &= O(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ R^V(q) &= O(2\pi/3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & -1/2 \end{pmatrix}, \\ R^V(q^2) &= O(4\pi/3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & -1/2 \end{pmatrix}. \end{aligned}$$

These matrices realize the group representation due to the relations $R^V(q^2) = (R^V(q))^2$ and $(R^V(q))^3 = R^V(q_0)$. The representation is faithful because the kernel of its homomorphism consists only of identity: $\text{Ker} R = q_0 \in C_3$.

Let us introduce the matrix

$$\hat{x} = x_i \cdot R^V(q_i) = \begin{pmatrix} x_0 + x_1 + x_2 & 0 & 0 \\ 0 & x_0 - 1/2(x_1 + x_2) & -\sqrt{3}/2(x_1 - x_2) \\ 0 & \sqrt{3}/2(x_1 - x_2) & x_0 - 1/2(x_1 + x_2) \end{pmatrix}. \quad (64)$$

The determinant of this matrix is

$$\text{Det}(\hat{x}) = x_0^3 + x_1^3 + x_2^3 - 3x_0x_1x_2. \quad (65)$$

where $R^{(1)}$ is the trivial representation, whereby each elements is mapped onto unit, *i.e.* for $R^{(1)}$ the kernel is the whole group, C_3 . For $R^{(2)}$ and $R^{(3)}$ the kernels can be identified with unit element, which means that they are faithful representations, isomorphic to C_3 .

Based on the character table one can obtain

$$\xi^V = (\xi^V(q_0), \xi^V(q), \xi^V(q^2)) = (3, 0, 0),$$

which demonstrates how the vector representation R^V decomposes in the irreducible representations $R^{(i)}$:

$$\xi^V = \xi^{(1)} + \xi^{(2)} + \xi^{(3)}$$

or

$$R^V = R^{(1)} \oplus R^{(2)} \oplus R^{(3)}.$$

The combinations of coordinates on which R^V acts irreducible are given below

$$\begin{pmatrix} z \\ \tilde{z} \\ \tilde{\tilde{z}} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & j & j^2 \\ 1 & j^2 & j \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 q \\ x_2 q^2 \end{pmatrix}.$$

4 Three dimensional theorem Pithagor and ternary complexification of \mathbb{R}^3

To go further here we must interpret some results from the ideas of the article [10, 11, 12, 14]. We would like to build the new numbers based on the \mathbb{C}_3 finite discrete group. For this let consider two basic elements, q_0, q_1 with the following constraints:

$$q_1 \cdot q_0 = q_0 \cdot q_1 = q_1, \quad q_1^3 = q_0, \quad (66)$$

In this case one can introduce a new element $q_2 = q_1^2 = q_1^{(-1)}$, *i.e.* $q_2 q_1 = q_1 q_2 = q_0$. From these three elements one can build a new field \mathbb{TC} :

$$\mathbb{TC} = \mathbb{R} \oplus \mathbb{R}q_1 \oplus \mathbb{R}q_1^2. \quad (67)$$

with the new numbers

$$z = x_0 q_0 + x_1 q_1 + x_2 q_2, \quad x_i \in \mathbb{R}, \quad i = 0, 1, 2, \quad (68)$$

which are the ternary generalization of the complex numbers.

Let define the operation of the conjugation:

$$\tilde{q}_1 = j q_1, \quad \tilde{\tilde{q}}_1 = j^2 q_1, \quad (69)$$

where $j = \exp(2i\pi)/3$. Since $q_2 = q_1^2$ one can easily get

$$\tilde{q}_2 = j^2 q_2, \quad \tilde{\tilde{q}}_2 = j q_2. \quad (70)$$

One can apply these two conjugation operations, respectively:

$$\begin{aligned} \tilde{z} &= x_0 q_0 + x_1 j q_1 + x_2 j^2 q_2, \\ \tilde{\tilde{z}} &= x_0 q_0 + x_1 j^2 q_1 + x_2 j q_2. \end{aligned} \quad (71)$$

Now one can introduce the cubic invariant form:

$$\langle z \rangle^3 = z \tilde{z} \tilde{\tilde{z}} = x_0^3 + x_1^3 + x_2^3 - 3x_0 x_1 x_2. \quad (72)$$

One can also easily check the following identity

$$\langle z_1 z_2 \rangle^3 = \langle z_1 \rangle^3 \langle z_2 \rangle^3, \quad (73)$$

which indicate about a group properties of these new \mathbb{TC} numbers. We suggest that this new Abelian group can be related with a ternary group?!

According to table of characters one can define two operations of the conjugations:

$$\bar{q}_1 = j q_1, \quad \bar{\bar{q}}_1 = j^2 q_1, \quad (74)$$

where $j = \exp(2i\pi)/3$. Since $q_2 = q_1^2$ we can easily obtain

$$\bar{q}_2 = j^2 q_2, \quad \bar{\bar{q}}_2 = j q_2. \quad (75)$$

These two conjugation operations can thus be applied, respectively:

$$\begin{aligned} \bar{\hat{z}} &= x_0 q_0 + x_1 j q_1 + x_2 j^2 q_2, \\ \bar{\bar{\hat{z}}} &= x_0 q_0 + x_1 j^2 q_1 + x_2 j q_2. \end{aligned} \quad (76)$$

We now introduce the cubic form:

$$\langle \hat{z} \rangle = \hat{z} \bar{\hat{z}} \bar{\bar{\hat{z}}} = x_0^3 + x_1^3 + x_2^3 - 3x_0 x_1 x_2, \quad (77)$$

The generators q and q^2 can be represented in the matrix form:

$$q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix}, \quad q^2 = \begin{pmatrix} 0 & 0 & j \\ 1 & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix} \quad (78)$$

where one can introduce the ternary transposition operations: $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ and $3 \rightarrow 2 \rightarrow 1 \rightarrow 3$. We now introduce the cubic form:

$$\langle \hat{z} \rangle = \hat{z} \bar{\hat{z}} \bar{\bar{\hat{z}}} = x_0^3 + x_1^3 + x_2^3 - 3x_0 x_1 x_2, \quad (79)$$

And also easily check the following relation:

$$\langle \hat{z}_1 \hat{z}_2 \rangle = \langle \hat{z}_1 \rangle \langle \hat{z}_2 \rangle, \quad (80)$$

which indicates the group properties of the TC numbers. More exactly, the unit TC numbers produce the Abelian ternary group. According to the ternary analogue of the Euler formula, the following ternary complex functions [10, 11, 12] can be constructed:

$$\Psi = \exp(q_1 \phi_1 + q_2 \phi_2), \quad \psi_1 = \exp(q_1 \phi_1), \quad \psi_2 = \exp(q_2 \phi_2), \quad (81)$$

where ϕ_i are the group parameters. For the functions ψ_i , $i = 0, 1, 2$, *i.e.* we have the following analogue of Euler, formula:

$$\begin{aligned} \Psi &= \exp(q_1 \phi + q_2 \phi_2) = f q_0 + g q_1 + h q_2, \\ \psi_1 &= \exp(q_1 \phi) = f_1 q_0 + g_1 q_1 + h_1 q_2, \\ \psi_2 &= \exp(q_2 \phi) = f_2 q_0 + h_2 q_1 + g_2 q_2, \end{aligned} \quad (82)$$

Consequently, we can now introduce the conjugation operations for these functions. For example, for ψ_1 we can get:

$$\begin{aligned} \bar{\psi}_1 &= \exp(\bar{q}_1 \phi) = \exp(j \cdot q_1 \phi) = f q_0 + j g q_1 + j^2 h q_2, \\ \bar{\bar{\psi}}_1 &= \exp(\bar{\bar{q}}_1 \phi) = \exp(j^2 \cdot q_1 \phi) = f_1 q_0 + j^2 g_1 q_1 + j h_1 q_2, \end{aligned} \quad (83)$$

with the following constraints:

$$\psi_1 \bar{\psi}_1 \bar{\bar{\psi}}_1 = \exp(q_1 \phi) \exp(j \cdot q_1 \phi) \exp(j^2 \cdot q_1 \phi) = q_0, \quad (84)$$

which gives us the following link between the functions, f, g, h :

$$f_1^3 + g_1^3 + h_1^3 - 3f_1 g_1 h_1 = 1. \quad (85)$$

This surface (see figure ??) is a ternary analogue of the S^1 circle and it is related with the ternary Abelian group, $TU(1)$.

The Euler formul:

$$\begin{aligned} z &= \rho \exp(\phi_1 q + \phi_2 q^2) = \rho \exp(\theta(q - q^2) + \phi(q + q^2)) \\ &= \rho(c(\phi_1, \phi_2) + s(\phi_1, \phi_2)q + t(\phi_1, \phi_2)q^2), \end{aligned} \quad (86)$$

where the Appel ternary trigonometric functions

$$\begin{aligned} c &= \frac{1}{3}(\exp(\phi_1 + \phi_2) + \exp(j\phi_1 + j^2\phi_2) + \exp(j^2\phi_1 + j\phi_2)) \\ s &= \frac{1}{3}(\exp(\phi_1 + \phi_2) + j^2 \exp(j\phi_1 + j^2\phi_2) + j \exp(j^2\phi_1 + j\phi_2)) \\ t &= \frac{1}{3}(\exp(\phi_1 + \phi_2) + j \exp(j\phi_1 + j^2\phi_2) + j^2 \exp(j^2\phi_1 + j\phi_2)) \end{aligned} \quad (87)$$

satisfy to the following equation:

$$c^3 + s^3 + t^3 - 3cst = 1. \quad (88)$$

There is also can be considered the ternary logarithmic function [14]:

$$\begin{aligned} \ln z &= (\ln z)_0 + (\ln z)_1 q + (\ln z)_2 q^2 \\ &= (\ln \rho)_0 + \phi_1 q + \phi_2 q^2, \end{aligned} \quad (89)$$

where

$$\begin{aligned} (\ln z)_0 &= \frac{1}{3} \ln(x_0^3 + x_1^3 + x_2^3 - 3x_0 x_1 x_2) \\ (\ln z)_1 &= \frac{1}{3} [\ln(x_0 + x_1 + x_2) + j^2 \ln(x_0 + jx_1 + j^2 x_2) + j \ln(x_0 + j^2 x_1 + jx_2)] \\ (\ln z)_2 &= \frac{1}{3} [\ln(x_0 + x_1 + x_2) + j \ln(x_0 + jx_1 + j^2 x_2) + j^2 \ln(x_0 + j^2 x_1 + jx_2)] \end{aligned} \quad (90)$$

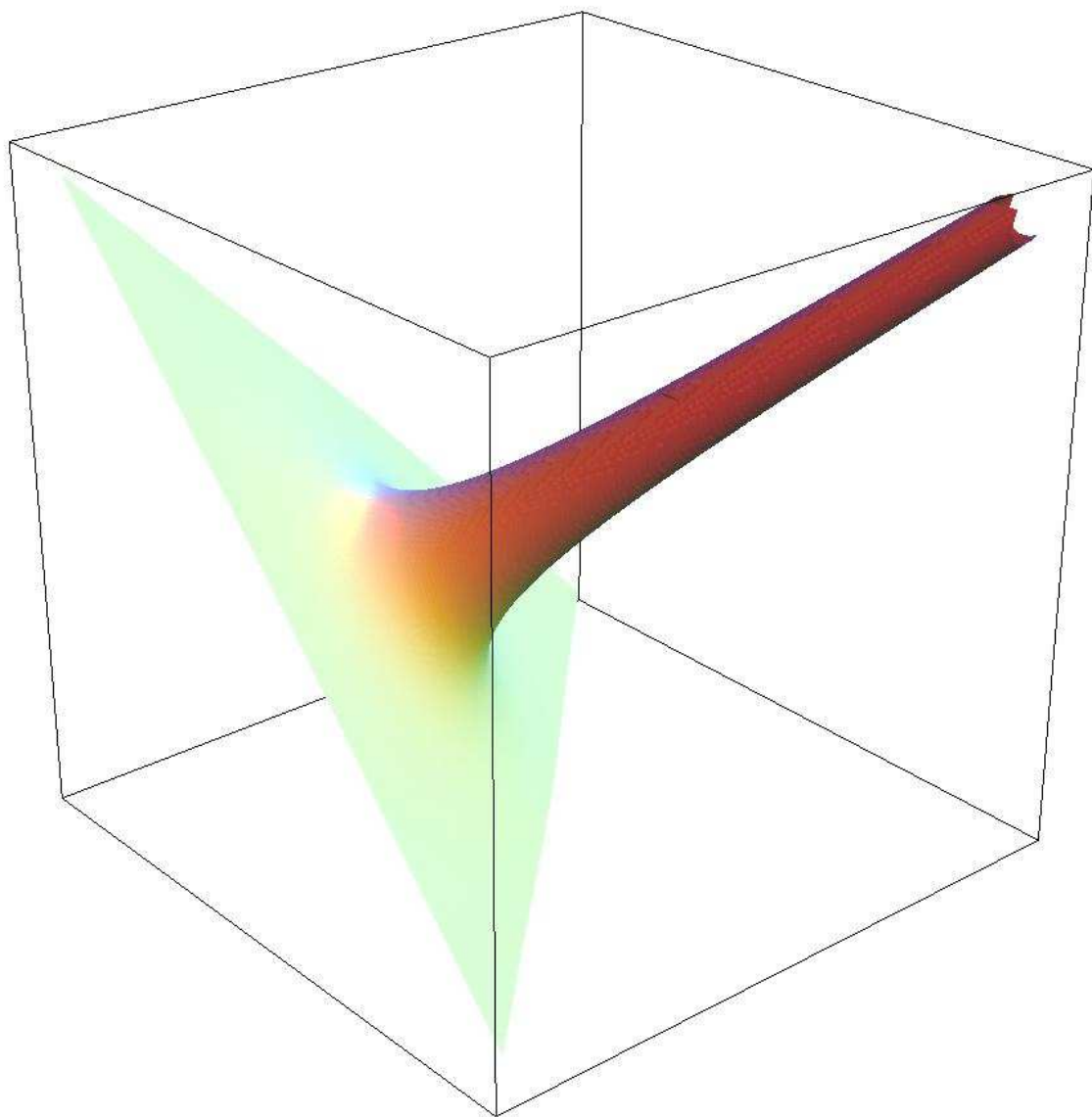


Figure 2: The surface for $f_1^3 + g_1^3 + h_1^3 - 3f_1g_1h_1 = 1$

For further use, note that for elements z, \tilde{z} and $\tilde{\tilde{z}}$ of the algebras¹ $\mathbb{T}_3\mathbb{C}, \tilde{\mathbb{T}}_3\mathbb{C}$ and $\tilde{\tilde{\mathbb{T}}}_3\mathbb{C}$ we have $\tilde{z} + \tilde{\tilde{z}} = 2x_0 - x_1q - x_2q^2 \in \mathbb{T}_3\mathbb{C}$, $\tilde{z}\tilde{\tilde{z}} = (x_0^2 - x_1x_2) + (x_2^2 - x_0x_1)q + (x_1^2 - x_2x_0)q^2 \in \mathbb{T}_3\mathbb{C}$. We also have

$$\begin{aligned} \|\cdot\|: \mathbb{T}_3\mathbb{C} \otimes \tilde{\mathbb{T}}_3\mathbb{C} \otimes \tilde{\tilde{\mathbb{T}}}_3\mathbb{C} &\rightarrow \mathbb{R}, \\ z \otimes \tilde{z} \otimes \tilde{\tilde{z}} &\mapsto \|z\|^3 = z\tilde{z}\tilde{\tilde{z}} = x_0^3 + x_1^3 + x_2^3 - 3x_0x_1x_2 \end{aligned}$$

. Thus, $\|z\| = 0$ if and only if z belongs to I_1 or to I_2 . A ternary complex number is called non-singular if $\|z\| \neq 0$. From now on we also denote $|z|$ the modulus of z .

It was proven in [?] that any non-singular ternary complex number $z \in \mathbb{T}_3\mathbb{C}$ can be written in the “polar form”:

$$z = \rho e^{\varphi_1 q + \varphi_2 q^2} = \rho e^{\theta(q - q^2) + \varphi(q + q^2)} \quad (91)$$

with $\rho = |z| = \sqrt[3]{x_0^3 + x_1^3 + x_2^3 - 3x_0x_1x_2} \in \mathbb{R}$, $\theta \in [0, 2\pi/\sqrt{3}[$, $\varphi \in \mathbb{R}$. The combinations $q - q^2$ and $q + q^2$ generate in the ternary space respectively compact and non-compact directions. Using $q + q^2 = 2K_0 + E_0$, we can rewrite in the form

$$z = \rho[m_0(\varphi_1, \varphi_2) + m_1(\varphi_1, \varphi_2)q + m_2(\varphi_1, \varphi_2)q^2] \quad (92)$$

Since for the product of two ternary complex numbers we have $\|zw\| = \|z\|\|w\|$ the set of unimodular ternary complex numbers preserves the cubic form [?, ?]. The continuous group of symmetry of the cubic surface $x_0^3 + x_1^3 + x_2^3 - 3x_0x_1x_2 = \rho^3$ is isomorphic to $SO(2) \times SO(1, 1)$. We denote the set of unimodular ternary complex numbers or the “ternary unit sphere” as $\mathbb{T}U(1) = \left\{ e^{(\theta + \varphi)q + (\varphi - \theta)q^2}, 0 \leq \theta < 2\pi/\sqrt{3}, \varphi \in \mathbb{R} \right\} \sim \mathbb{T}S^1$.

From the above figure one can see, that this surface approaches asymptotically the plane $x_0 + x_1 + x_2 = 0$ and the line $x_0 = x_1 = x_2$ orthogonal to it. In $\mathbb{T}_3\mathbb{C}$ they correspond to the ideals I_2 and I_1 , respectively. The latter line will be called the “trisectrice”.

Let give the Pithagorean theorem through the differential 2-forms. One can construct the inner metric of this surface for the general case $\rho \neq 0$. Introduce $a = x_0 + x_1 + x_2$ and parametrise a point on the circle of radius r around the trisectrice by its polar coordinates (r, θ) . The surface

$$x_0^3 + x_1^3 + x_2^3 - 3x_0x_1x_2 = \rho^3 \quad (93)$$

in these coordinates has the simple equation

$$ar^2 = \rho^3. \quad (94)$$

It can be shown that for this cubic surface we can choose a parametrization, $g(a, \theta) : \mathbb{R}^2 \rightarrow \Sigma$ for a point $M(x_0, x_1, x_2)$:

$$g(a, \theta) = (x_0(a, \theta), x_1(a, \theta), x_2(a, \theta)) \quad (95)$$

¹If we complexify the ternary complex numbers $\mathbb{T}_3\mathbb{C}^{\mathbb{C}} = \mathbb{T}_3\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, the three above copies become identical and \sim is an automorphism.

where

$$\begin{aligned}
x_0(a, \theta) &= \frac{a}{3} - \frac{2}{3} \frac{\rho^{3/2}}{\sqrt{a}} \cos \theta, \\
x_1(a, \theta) &= \frac{a}{3} + \frac{1}{3} \frac{\rho^{3/2}}{\sqrt{a}} (\cos \theta + \sqrt{3} \sin \theta), \\
x_2(a, \theta) &= \frac{a}{3} + \frac{1}{3} \frac{\rho^{3/2}}{\sqrt{a}} (\cos \theta - \sqrt{3} \sin \theta).
\end{aligned} \tag{96}$$

Now one can find the tangent vectors to the surface $\Sigma \subset \mathbb{R}^3$ in the point $x_0(a, \theta), x_1(a, \theta), x_2(a, \theta)$

$$\begin{aligned}
\frac{\partial g}{\partial a} &= \left(\frac{\partial x_0}{\partial a}, \frac{\partial x_1}{\partial a}, \frac{\partial x_2}{\partial a} \right) \\
\frac{\partial g}{\partial \theta} &= \left(\frac{\partial x_0}{\partial \theta}, \frac{\partial x_1}{\partial \theta}, \frac{\partial x_2}{\partial \theta} \right)
\end{aligned} \tag{97}$$

or

$$\begin{aligned}
\frac{\partial x_0}{\partial a} &= \frac{1}{3} + \frac{1}{3} \frac{\rho^{3/2}}{a^{3/2}} \cos \theta \\
\frac{\partial x_1}{\partial a} &= \frac{1}{3} - \frac{1}{6} \frac{\rho^{3/2}}{a^{3/2}} (\cos \theta + \sqrt{3} \sin \theta) \\
\frac{\partial x_2}{\partial a} &= \frac{1}{3} - \frac{1}{6} \frac{\rho^{3/2}}{a^{3/2}} (\cos \theta - \sqrt{3} \sin \theta)
\end{aligned} \tag{98}$$

and

$$\begin{aligned}
\frac{\partial x_0}{\partial \theta} &= \frac{2}{3} \frac{\rho^{3/2}}{\sqrt{a}} \sin \theta \\
\frac{\partial x_1}{\partial \theta} &= \frac{1}{3} \frac{\rho^{3/2}}{\sqrt{a}} (-\sin \theta + \sqrt{3} \cos \theta) \\
\frac{\partial x_2}{\partial \theta} &= \frac{1}{3} \frac{\rho^{3/2}}{\sqrt{a}} (-\sin \theta - \sqrt{3} \cos \theta)
\end{aligned} \tag{99}$$

These two tangent vectors allow to calculate the area of the parallelogram based on them:

$$J_{123} = \begin{vmatrix} \frac{\partial x_0}{\partial a} & \frac{\partial x_1}{\partial a} & \frac{\partial x_2}{\partial a} \\ \frac{\partial x_0}{\partial \theta} & \frac{\partial x_1}{\partial \theta} & \frac{\partial x_2}{\partial \theta} \\ 1 & 1 & 1 \end{vmatrix}. \tag{100}$$

Geometrically, the differential forms $dx_0 \wedge dx_1$, $dx_1 \wedge dx_2$, $dx_2 \wedge dx_0$ are the areas of the parallelograms spanned by the vectors $\frac{\partial g}{\partial a}$ and $\frac{\partial g}{\partial \theta}$ projected onto the $dx_0 - dx_1$, $dx_1 - dx_2$, $dx_2 - dx_0$ planes, respectively. This gives

$$dx_k \wedge dx_l = J_{kl} da d\theta, \quad k, l = 0, 1, 2, \quad (101)$$

where the Jacobians are

$$J_{01} = \begin{vmatrix} \frac{\partial x_k}{\partial a} & \frac{\partial x_l}{\partial a} & 0 \\ \frac{\partial x_k}{\partial \theta} & \frac{\partial x_l}{\partial \theta} & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad (102)$$

$$J_{12} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \frac{\partial x_k}{\partial a} & \frac{\partial x_l}{\partial a} \\ 0 & \frac{\partial x_k}{\partial \theta} & \frac{\partial x_l}{\partial \theta} \end{vmatrix} \quad (103)$$

$$J_{20} = \begin{vmatrix} \frac{\partial x_k}{\partial a} & 0 & \frac{\partial x_l}{\partial a} \\ 0 & 1 & 0 \\ \frac{\partial x_k}{\partial \theta} & 0 & \frac{\partial x_l}{\partial \theta} \end{vmatrix} \quad (104)$$

Now one can see the geometrical meaning of J_{01} , J_{12} , J_{20} and J_{012} and to get the ternary analog of the Pithagorean theorem:

$$J_{01}^3 + J_{12}^3 + J_{20}^3 - 3J_{01}J_{12}J_{20} = J_{012}^3 = \frac{1}{3\sqrt{3}} \frac{\rho^6}{a^3}. \quad (105)$$

From parallelogram Pithagorean theorem one can easily come to the tetrahedron Pithagorean theorem:

$$S_A^3 + S_B^3 + S_C^3 - 3S_AS_BS_C = S_D^3, \quad (106)$$

where we have for $S_{...}$ four triangle faces of the tetrahedron.

The $TSO(2) \times TSO(1,1)$ group of transformations are generated by the ternary-sine functions. In particular, in the special case where $\varphi = 0$ the transformation in the compact direction is a rotation to the angle $\sqrt{3}\theta$ and for $\theta = 0$ we have the dilatation in the non-compact direction

$$\begin{aligned} \varphi = 0 : & \begin{cases} x_0 + x_1 + x_2 \rightarrow x_0 + x_1 + x_2 \\ x_0 + jx_1 + j^2x_2 \rightarrow e^{i\sqrt{3}\theta}(x_0 + jx_1 + j^2x_2) \end{cases} , \\ \theta = 0 : & \begin{cases} x_0 + x_1 + x_2 \rightarrow e^{2\varphi}(x_0 + x_1 + x_2) \\ x_0 + jx_1 + j^2x_2 \rightarrow e^{-\varphi}(x_0 + jx_1 + j^2x_2) \end{cases} . \end{aligned}$$

Let us consider now the discrete transformation preserving the modulus $\|z\|$ of non-singular ternary complex numbers:

$$z = \rho e^{\varphi_1 q + \varphi_2 q^2} \rightarrow \bar{z} = \frac{\tilde{z} \tilde{\bar{z}}}{\|z\|} = \rho e^{-\varphi_1 q - \varphi_2 q^2}. \quad (107)$$

We are going to investigate new aspects of the ternary complex analysis based on the “complexification” of R^3 space. The use of the cyclic C_3 group for this purpose is a natural generalization of the similar procedure for the $C_2 = Z_2$ group in two dimensions. It is known that the complexification of R^2 allows to introduce the new geometrical objects - the Riemannian surfaces. The Riemannian surfaces are defined as a pair (M, C) , where M is a connected two-dimensional manifold and C is a complex structure on M . Well-known examples of Riemann surfaces are the complex plane- C , Riemann sphere - $CP^1 : C \cup \infty$ and complex tori- $T = C/\Gamma$, $\Gamma := n\lambda_1 + m\lambda_2 : n, m \in Z, \lambda_{1,2} \in C$.

Let us introduce the complex valued functions $f(x_0, x_1) = a(x_0, x_1) + ib(x_0, x_1)$ in an open subset $U \subset C$.

The harmonic functions $a(x_0, x_1)$ and $b(x_0, x_1)$ satisfy the Laplace equations:

$$\begin{aligned} \frac{\partial^2 a}{\partial z \partial \bar{z}} dz \wedge d\bar{z} &= \frac{1}{2i} \left(\frac{\partial^2 a}{\partial x_0^2} + \frac{\partial^2 a}{\partial x_1^2} \right) dx \wedge dy = 0, \\ \frac{\partial^2 b}{\partial z \partial \bar{z}} dz \wedge d\bar{z} &= \frac{1}{2i} \left(\frac{\partial^2 b}{\partial x_0^2} + \frac{\partial^2 b}{\partial x_1^2} \right) dx \wedge dy = 0. \end{aligned} \quad (108)$$

These equations are invariant under the $SO(2)$ transformations, which is a consequence of the symmetry of the $U(1)$ bilinear form $\{z\bar{z} = (x_0 + ix_1)(x_0 - ix_1) = 1\} = S^1$ under the phase multiplication:

$$z \rightarrow \exp\{i\alpha\}z, \quad \bar{z} \rightarrow \exp\{-i\alpha\}\bar{z}.$$

According to Dirac one can make the square root from Laplace equation:

$$\sigma_1 \frac{\partial \psi}{\partial x_0} + \sigma_2 \frac{\partial \psi}{\partial x_1} = 0, \quad (109)$$

where a field ψ is two-dimensional spinor

$$\psi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad (110)$$

and where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (111)$$

are the famous Pauli matrices:

$$\sigma_m \sigma_n + \sigma_n \sigma_m = 2\delta_{mn}, \quad m, n = 1, 2, \quad (112)$$

which with σ_3 and σ_0 are

$$\begin{aligned}\sigma_3 &= i\sigma_1\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \sigma_0 &= \sigma_i^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = 1, 2, 3.\end{aligned}\tag{113}$$

Thus on the complex plane we have the following Dirac relation:

$$(\sigma_1 \frac{\partial}{\partial x_0} + \sigma_2 \frac{\partial}{\partial x_1})^2 = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2}\tag{114}$$

Due to properties of all σ_i , $i = 1, 2, 3$ matrices the similar link remains valid in $D = 3$:

$$(\sigma_1 \frac{\partial}{\partial x_0} + \sigma_2 \frac{\partial}{\partial x_1} + \sigma_3 \frac{\partial}{\partial x_2})^2 = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.\tag{115}$$

In $D = 4$ one can get similar link if take into account the conjugation properties of quaternions:

$$\begin{aligned}& (\sigma_0 \frac{\partial}{\partial x_0} + i\sigma_1 \frac{\partial}{\partial x_1} + i\sigma_2 \frac{\partial}{\partial x_2} + i\sigma_3 \frac{\partial}{\partial x_3}) \cdot (\sigma_0 \frac{\partial}{\partial x_0} - i\sigma_1 \frac{\partial}{\partial x_1} - i\sigma_2 \frac{\partial}{\partial x_2} - i\sigma_3 \frac{\partial}{\partial x_3}) \\ &= \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.\end{aligned}\tag{116}$$

Note that through the Pauli matrices:

$$\sigma_0, \mathbf{i}\sigma_1, \mathbf{i}\sigma_2, \mathbf{i}\sigma_3,\tag{117}$$

there is the matrix realization of quaternions

$$q = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3, \quad q \in \mathbb{H},\tag{118}$$

which produce over \mathbb{R} a 4-dimensional norm division algebra where appears the third imaginary unit $e_3 = e_1 e_2 \equiv \mathbf{k}$.

The set of Pauli matrices produces the Clifford algebra

$$\begin{aligned}& \sigma_0 \\ & \sigma_1, \sigma_2 \\ & \sigma_1 \sigma_2\end{aligned}\tag{119}$$

and solution of linearized Dirac equation one should look for through the spinor fields:

$$\psi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}\tag{120}$$

Thus in 2- dimensional space one can introduce the spin structure, what was related to the complexification of \mathbb{R}^2 . Dirac made the square root from the relativistic Klein-Gordon equation extending the binary Clifford algebra into four dimensional space-time:

$$\gamma_m \gamma_n + \gamma_n \gamma_m = 2g_{mn}, \quad m, n = 0, 1, 2, 3. \quad (121)$$

where

$$\begin{aligned} \gamma_0 &= \sigma_1 \otimes \sigma_0 \\ \gamma_1 &= \sigma_3 \otimes \sigma_0 \\ \gamma_2 &= \sigma_2 \otimes \sigma_1 \\ \gamma_3 &= \sigma_2 \otimes \sigma_3 \end{aligned} \quad (122)$$

In the relativistic Dirac equation one should consider already the bispinors (ψ_1, ψ_2) which already have got in addition to spin structure a new geometrical structure related to the discovery antiparticle states. each new structure will appear in $\mathbb{R}^{6,8,\dots}$ space.

Now consider the \mathbb{C}_3 - holomorphicity.

Let us consider the function

$$F(z, \tilde{z}, \tilde{\tilde{z}}) = f_0(x_0, x_1, x_2) + f_1(x_0, x_1, x_2)q + f_2(x_0, x_1, x_2)q^2 \quad (123)$$

For the C_3 holomorphicity we have two types:

- 1. For the first type of holomorphicity function $F(z, \tilde{z}, \tilde{\tilde{z}})$ we have the following two conditions:

$$\frac{\partial F(z, \tilde{z}, \tilde{\tilde{z}})}{\partial \tilde{z}} = \frac{\partial F(z, \tilde{z}, \tilde{\tilde{z}})}{\partial \tilde{\tilde{z}}} = 0. \quad (124)$$

- 2. For the second type of holomorphicity function $F(z, \tilde{z}, \tilde{\tilde{z}})$ we can take just one condition:

$$\frac{\partial F(z, \tilde{z}, \tilde{\tilde{z}})}{\partial z} = 0. \quad (125)$$

$$\begin{aligned} \begin{pmatrix} \partial_{z_0} \\ \partial_{z_1} \\ \partial_{z_2} \end{pmatrix} &= \left\| \frac{\partial x_p}{\partial z_r} \right\| \begin{pmatrix} \partial_0 \\ \partial_1 \\ \partial_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial x_0}{\partial z} & \frac{\partial x_1}{\partial z} & \frac{\partial x_2}{\partial z} \\ \frac{\partial x_0}{\partial z_1} & \frac{\partial x_1}{\partial z_1} & \frac{\partial x_2}{\partial z_1} \\ \frac{\partial x_0}{\partial z_2} & \frac{\partial x_1}{\partial z_2} & \frac{\partial x_2}{\partial z_2} \end{pmatrix} \begin{pmatrix} \partial_0 \\ \partial_1 \\ \partial_2 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 1 & q^2 & q \\ 1 & j^2 q^2 & jq \\ 1 & jq^2 & j^2 q \end{pmatrix} \begin{pmatrix} \partial_0 \\ \partial_1 \\ \partial_2 \end{pmatrix} \end{aligned} \quad (126)$$

Here we used

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ q^2 & j^2 q^2 & j q^2 \\ q & j q & j^2 q^2 \end{pmatrix} \begin{pmatrix} z \\ \tilde{z} \\ \tilde{\tilde{z}} \end{pmatrix} \quad (127)$$

More shortly:

$$\partial_{z_p} = J_{pr} \partial_r \quad (128)$$

where

$$J_{pr} = \left\| \frac{\partial x_p}{\partial z_r} \right\| = \frac{1}{3} \begin{pmatrix} 1 & q^2 & q \\ 1 & j^2 q^2 & j q \\ 1 & j q^2 & j^2 q \end{pmatrix} \quad (129)$$

is Jacobian. We took some useful notations:

$\partial_{z_p} = \frac{\partial}{\partial z_p}$, and $\partial_p = \frac{\partial}{\partial x_p}$, $z_1 \equiv \tilde{z}$, $z_2 \equiv \tilde{\tilde{z}}$, $z_3 \equiv \tilde{\tilde{\tilde{z}}}$, and $p, r = 0, 1, 2$.

The inverse parities are the following:

$$\begin{pmatrix} \partial_0 \\ \partial_1 \\ \partial_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ q & j q & j^2 q \\ q^2 & j^2 q^2 & j q^2 \end{pmatrix} \begin{pmatrix} \partial_{z_0} \\ \partial_{z_1} \\ \partial_{z_2} \end{pmatrix} \quad (130)$$

As result, for all three derivatives in we can give the following expressions:

$$\begin{aligned} \partial_z F &= \frac{1}{3} (\partial_0 + q^2 \partial_1 + q \partial_2) (f_0 + q f_1 + q^2 f_2) \\ &= \frac{1}{3} (\partial_0 f_0 + \partial_1 f_1 + \partial_2 f_2) \\ &+ \frac{1}{3} (\partial_2 f_0 + \partial_0 f_1 + \partial_1 f_2) q \\ &+ \frac{1}{3} (\partial_1 f_0 + \partial_2 f_1 + \partial_0 f_2) q^2, \end{aligned} \quad (131)$$

$$\begin{aligned} \partial_{z_1} F &= \frac{1}{3} (\partial_0 + j^2 q^2 \partial_1 + j q \partial_2) (f_0 + q f_1 + q^2 f_2) \\ &= \frac{1}{3} (\partial_0 f_0 + j^2 \partial_1 f_1 + j \partial_2 f_2) \\ &+ \frac{1}{3} (j \partial_2 f_0 + \partial_0 f_1 + j^2 \partial_1 f_2) q \\ &+ \frac{1}{3} (j^2 \partial_1 f_0 + j \partial_2 f_1 + \partial_0 f_2) q^2, \end{aligned} \quad (132)$$

$$\begin{aligned}
\partial_{z_2} F &= \frac{1}{3}(\partial_0 + jq^2\partial_1 + j^2q\partial_2)(f_0 + qf_1 + q^2 + f_2) \\
&= \frac{1}{3}(\partial_0 f_0 + j\partial_1 f_1 + j^2\partial_2 f_2) \\
&+ \frac{1}{3}(j^2\partial_2 f_0 + \partial_0 f_1 + j\partial_1 f_2)q \\
&+ \frac{1}{3}(j\partial_1 f_0 + j^2\partial_2 f_1 + \partial_0 f_2)q^2.
\end{aligned} \tag{133}$$

The first type constraints $\partial_{z_1} F = \partial_{z_2} = 0$ give us the following differential equations:

$$\begin{aligned}
I. \quad & \partial_0 f_0 + j^2\partial_1 f_1 + j\partial_2 f_2 = 0, \\
II. \quad & j\partial_2 f_0 + \partial_0 f_1 + j^2\partial_1 f_2 = 0, \\
III. \quad & j^2\partial_1 f_0 + j\partial_2 f_1 + \partial_0 f_2 = 0
\end{aligned} \tag{134}$$

and

$$\begin{aligned}
IV. \quad & \partial_0 f_0 + j\partial_1 f_1 + j^2\partial_2 f_2 = 0, \\
V. \quad & j^2\partial_2 f_0 + \partial_0 f_1 + j\partial_1 f_2 = 0, \\
VI. \quad & j\partial_1 f_0 + j^2\partial_2 f_1 + \partial_0 f_2 = 0.
\end{aligned} \tag{135}$$

Let solve the system of these six equations. For this let take the first equations, I and IV, from both system, multiply the equation I on the j^2 and the equation IV on j :

$$\begin{aligned}
j^2\partial_0 f_0 + j\partial_1 f_1 + \partial_2 f_2 &= 0, \\
j\partial_0 f_0 + j^2\partial_1 f_1 + \partial_2 f_2 &= 0.
\end{aligned} \tag{136}$$

Having taken the difference of the equations one can get the Cauchy-Riemann parity:

$$\partial_0 f_0 = \partial_1 f_1 \tag{137}$$

Similarly, one can get the full system of the linear differential equations:

$$\begin{aligned}
\partial_0 f_0 &= \partial_1 f_1 = \partial_2 f_2 \\
\partial_1 f_0 &= \partial_2 f_1 = \partial_0 f_2 \\
\partial_2 f_0 &= \partial_0 f_1 = \partial_1 f_2
\end{aligned} \tag{138}$$

These equations give the definition of ternary harmonics functions, the analogue of the Caushi-Riemann (Darbu-Euleur) definition for holomorphic functions in the ordinary binary case.

From these equations one can get also that the three harmonics functions, $f_0(x_0, x_1, x_2)$, $f_1(x_0, x_1, x_2)$, $f_2(x_0, x_1, x_2)$, defined from the holomorphic function

$$F(z) = f_0(x_0, x_1, x_2) + qf_1(x_0, x_1, x_2) + q^2f_2(x_0, x_1, x_2) \quad (139)$$

are satisfied to the cubic Laplace equations:

$$\partial_0^3 f_p + \partial_1^3 f_p + \partial_2^3 f_p - 3\partial_0\partial_1\partial_2 f_p = 0, \quad p = 0, 1, 2. \quad (140)$$

Let show this for the harmonics function $f_0(x, y, u)$. For this one should build the next combinations:

$$\begin{aligned} \partial_0^3 f_0 &= \partial_0^2 \partial_1 f_1 = \partial_0^2 \partial_1 f_2 \\ \partial_1^3 f_0 &= \partial_1^2 \partial_2 f_1 = \partial_1^2 \partial_0 f_2 \\ \partial_2^3 f_0 &= \partial_2^2 \partial_0 f_1 = \partial_2^2 \partial_1 f_2 \end{aligned} \quad (141)$$

and

$$\begin{aligned} \partial_0\partial_1\partial_2 f_0 &= \partial_2^2 \partial_0 f_1 \\ \partial_0\partial_1\partial_2 f_0 &= \partial_1^2 \partial_2 f_1 \\ \partial_0\partial_1\partial_2 f_0 &= \partial_0^2 \partial_1 f_1 \end{aligned} \quad (142)$$

Compare the two systems of differential equations for $f_0(x_0, x_1, x_2)$ one can get the ternary Laplace equation. Similarly, one can get such equations for harmonics functions $f_1(x_0, x_1, x_2)$ and $f_2(x_0, x_1, x_2)$.

Thus the ternary holomorphic analysis in \mathbb{R}^3 leads to ternary harmonic functions: $f(z) = f_0(x_0, x_1, x_2) + qf_1(x_0, x_1, x_2) + q^2f_2(x_0, x_1, x_2)$ which satisfied to cubic differential equations:

$$\frac{\partial^3 f_i}{\partial x_0^3} + \frac{\partial^3 f_i}{\partial x_1^3} + \frac{\partial^3 f_i}{\partial x_2^3} - 3\frac{\partial^3 f_i}{\partial x_0\partial x_1\partial x_2} = 0 \quad (143)$$

Let introduce the 3×3 matrices:

$$Q_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j^2 \\ j & 0 & 0 \end{pmatrix} \quad (144)$$

These matrices satisfy to some remarkable relations:

$$Q_a Q_b Q_c + Q_b Q_c Q_a + Q_c Q_a Q_b = 3\eta_{abc} E_0 \quad (145)$$

with

$$\begin{aligned}
\eta_{111} &= \eta_{222} = \eta_{333} = 1 \\
\eta_{123} &= \eta_{231} = \eta_{312} = j \\
\eta_{321} &= \eta_{213} = \eta_{132} = j^2
\end{aligned}
\tag{146}$$

where $j = \exp(2\pi/3)$.

Using these matrices one can get the ternary Dirac equation:

$$Q_1 \frac{\partial \Psi}{\partial x_0} + Q_2 \frac{\partial \Psi}{\partial x_1} + Q_3 \frac{\partial \Psi}{\partial x_2} = 0, \tag{147}$$

where

$$\Psi = (\psi_1, \psi_2, \psi_3), \tag{148}$$

is triplet of the wave functions, *i.e.* we introduced the ternary spin structure in \mathbb{R}^3 . The next ternary structures can appear in $\mathbb{R}^{6,9,12,\dots}$ spaces.

In order to diagonalize this equation we must act three times with the same operator and we will get the cubic differential equation satisfied by each component ψ_p , $p = 1, 2, 3$.

5 The symmetry of the cubic forms

The complex number theory is a seminal field in mathematics having many applications to geometry, group theory, algebra and also to the classical and quantum physics. Geometrically, it is based on the complexification of the R^2 plane. The existence of similar structures in higher dimensional spaces is interesting for phenomenological applications.

It is easily to check the following relation:

$$\langle \hat{z}_1 \hat{z}_2 \rangle = \langle \hat{z}_1 \rangle \langle \hat{z}_2 \rangle, \quad (149)$$

which indicates the group properties of the \mathbb{TC} numbers. More exactly, the unit \mathbb{TC} numbers produce the Abelian ternary group. According to the ternary analogue of the Euler formula, the following "unitary" ternary $TU(1)$ group can be constructed:

$$U = \exp(q\alpha + q^2\beta) \quad (150)$$

where α, β are the group parameters. The 'unitarity' condition is:

$$U \cdot \tilde{U} \cdot \tilde{\tilde{U}} = \hat{1}, \quad (151)$$

where

$$\begin{aligned} \tilde{U} &= \exp(jq\alpha + j^2q^2\beta) \\ \tilde{\tilde{U}} &= \exp(j^2q\alpha + jq^2\beta), \end{aligned} \quad (152)$$

Similarly to the binary case, when for the $U(1)$ Abelian group one can find the form of $SO(2)$ group, there also exists such a correspondence. For simplicity, take $\beta = 0$. Then

$$\begin{aligned} U \rightarrow O &= \begin{pmatrix} c & s & t \\ t & c & s \\ s & t & c \end{pmatrix} \\ \tilde{U} \rightarrow \tilde{O} &= \begin{pmatrix} c & js & j^2t \\ j^2t & c & js \\ js & j^2t & c \end{pmatrix} \\ \tilde{\tilde{U}} \rightarrow \tilde{\tilde{O}} &= \begin{pmatrix} c & j^2s & jt \\ jt & c & j^2s \\ j^2s & jt & c \end{pmatrix} \end{aligned} \quad (153)$$

where

$$O \cdot \tilde{O} \cdot \tilde{\tilde{O}} = c^3 + s^3 + t^3 - 3cst \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (154)$$

Let us find

$$(x', y', u')^t = O_{Si} \cdot (x, y, u)^t. \quad (155)$$

where

$$O_S = \exp\{\alpha q_1 + \beta q_1^2\} = O_{S1} O_{S2} \quad (156)$$

and

$$O_{S1} = \exp\{\alpha q_1\}, \quad O_{S2} = \exp\{\beta q_1^2\}, \quad (157)$$

respectively.

The generators q_1 and q_1^2 can be represented in the matrix form:

$$q_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad q_1^2 = q_4 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (158)$$

Let find the eigenvalues

$$\det\{\alpha q_1 + \beta q_1^2 - \lambda E\} = \det \begin{pmatrix} -\lambda & \alpha & \beta \\ \beta & -\lambda & \alpha \\ \alpha & \beta & -\lambda \end{pmatrix} = -\lambda^3 + \alpha^3 + \beta^3 - 3\lambda\alpha\beta = 0 \quad (159)$$

So, we have the following three eigenvalues:

$$\lambda_1 = \alpha + \beta, \quad \lambda_2 = j\alpha + j^2\beta, \quad \lambda_3 = j^2\alpha + j\beta. \quad (160)$$

$$O_S = S S^{-1} \exp\{\alpha q_1 + \beta q_1^2\} S S^{-1} = S \exp\{S^{-1}(\alpha q_1 + \beta q_1^2)S\} S^{-1}, \quad (161)$$

$$S = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & j & j^2 \\ 1 & j^2 & j \end{pmatrix}, \quad S^{-1} = S^+ = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & j^2 & j \\ 1 & j & j^2 \end{pmatrix}, \quad (162)$$

$$\begin{aligned}
& S \exp\{\alpha S^{-1}(\alpha q_1 + \beta q_1^2)S\}S^{-1} = S \exp\left\{\begin{pmatrix} \alpha + \beta & 0 & 0 \\ 0 & j\alpha + j^2\beta & 0 \\ 0 & 0 & j^2\alpha + j\beta \end{pmatrix}\right\}S^{-1} \\
& = S \exp\left\{\alpha \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix}\right\}S^{-1}S \exp\left\{\beta \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & j \end{pmatrix}\right\}S^{-1} \\
& = S \begin{pmatrix} c_1(\alpha) + s_1(\alpha) + t_1(\alpha) & 0 & 0 \\ 0 & c_1(\alpha) + js_1(\alpha) + j^2t_1(\alpha) & 0 \\ 0 & 0 & c_1(\alpha) + j^2s_1(\alpha) + jt_1(\alpha) \end{pmatrix} S^{-1} \\
& \cdot S \begin{pmatrix} c_2(\beta) + s_2(\beta) + t_2(\beta) & 0 & 0 \\ 0 & c_2(\beta) + j^2s_2(\beta) + jt_2(\beta) & 0 \\ 0 & 0 & c_2(\beta) + js_2(\beta) + j^2t_2(\beta) \end{pmatrix} S^{-1} \\
& = \begin{pmatrix} c_1(\alpha) & s_1(\alpha) & t_1(\alpha) \\ t_1(\alpha) & c_1(\alpha) & s_1(\alpha) \\ s_1(\alpha) & t_1(\alpha) & c_1(\alpha) \end{pmatrix} \begin{pmatrix} c_2(\beta) & t_2(\beta) & s_2(\beta) \\ s_2(\beta) & c_2(\beta) & t_2(\beta) \\ t_2(\beta) & s_2(\beta) & c_2(\beta) \end{pmatrix}
\end{aligned} \tag{163}$$

Let us consider two limit cases:

- 1 case: $\alpha = -\beta$
- 2 case: $\alpha = \beta$

The case 1 is related to the binary orthogonal symmetry of the cubic surface and cubic forms. This orthogonal symmetry is in the plane which orthogonol to the direction of “trisectriss”.

$$\begin{aligned}
O_S &= \exp\{\alpha(q_1 - q_1^2)\} = S \exp\{S^{-1}(\alpha(q_1 - q_1^2))S\}S^{-1} \\
&= S \exp\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha(j - j^2) & 0 \\ 0 & 0 & \alpha(j^2 - j) \end{pmatrix}\right\}S^{-1} \\
&= S \begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp\{\alpha(j - j^2)\} & 0 \\ 0 & 0 & \exp\{\alpha(j^2 - j)\} \end{pmatrix} S^{-1} \\
&= \frac{1}{3} \begin{pmatrix} 1 + e^{\{i\phi\}} + e^{\{-i\phi\}} & 1 + j^2e^{\{i\phi\}} + je^{\{-i\phi\}} & 1 + je^{\{i\phi\}} + j^2e^{\{-i\phi\}} \\ 1 + je^{\{i\phi\}} + j^2e^{\{-i\phi\}} & 1 + e^{\{i\phi\}} + e^{\{-i\phi\}} & 1 + j^2e^{\{i\phi\}} + e^{\{-i\phi\}} \\ 1 + j^2e^{\{i\phi\}} + je^{\{-i\phi\}} & 1 + je^{\{i\phi\}} + j^2e^{\{-i\phi\}} & 1 + e^{\{i\phi\}} + e^{\{-i\phi\}} \end{pmatrix}
\end{aligned} \tag{164}$$

where $j - j^2 = \sqrt{3}\mathbf{i}$, $\phi = \sqrt{3}\alpha$.

Thus

$$O_S = \begin{pmatrix} c_0 & s_0 & t_0 \\ t_0 & c_0 & s_0 \\ s_0 & t_0 & c_0 \end{pmatrix}, \quad (165)$$

where we have the particular choice for the functions, c , s , t :

$$\begin{aligned} c_0 &= \frac{1}{3}(1 + e^{\{i\phi\}} + e^{\{-i\phi\}}) = \frac{1}{3}(1 + 2\cos(\phi)) \\ s_0 &= \frac{1}{3}(1 + j^2 e^{\{i\phi\}} + j e^{\{-i\phi\}}) = \frac{1}{3}(1 + 2\cos(\phi + \frac{2\pi}{3})), \\ t_0 &= \frac{1}{3}(1 + j e^{\{i\phi\}} + j^2 e^{\{-i\phi\}}) = (\frac{1}{3}1 + 2\cos(\phi - \frac{2\pi}{3})). \end{aligned} \quad (163)$$

One can check that $c_0^3 + s_0^3 + t_0^3 - 3c_0s_0t_0 = 1$. But these transformations are also binary orthogonal transformations. It means that the matrices

$$O = \begin{pmatrix} c_0 & s_0 & t_0 \\ t_0 & c_0 & s_0 \\ s_0 & t_0 & c_0 \end{pmatrix}, \quad (164)$$

and

$$O^t = \begin{pmatrix} c_0 & t_0 & s_0 \\ s_0 & c_0 & t_0 \\ t_0 & s_0 & c_0 \end{pmatrix}, \quad (165)$$

satisfy to condition $OO^t = O^tO = 1$, what is equivalent to the additional two equations:

$$\begin{aligned} c_0^2 + s_0^2 + t_0^2 &= 1 \\ c_0s_0 + s_0t_0 + t_0c_0 &= 0, \end{aligned} \quad (164)$$

what in our case can be easily checked. Thus in the case 1 the ternary symmetry coincides with the orthogonal binary symmetry $SO(2)$.

In the case 2, $\alpha = \beta$, the ternary symmetry coincides with the other binary symmetry.

$$\begin{aligned}
O_S &= \exp\{\alpha(q_1 + q_1^2)\} = S \exp\{S^{-1}(\alpha(q_1 + q_1^2))S\}S^{-1} \\
&= S \exp\left\{\alpha \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right\}S^{-1} \\
&= S \begin{pmatrix} \exp\{2\alpha\} & 0 & 0 \\ 0 & \exp\{-\alpha\} & 0 \\ 0 & 0 & \exp\{-\alpha\} \end{pmatrix} S^{-1} \\
&= \frac{1}{3} \begin{pmatrix} e^{\{2\alpha\}} + 2e^{\{-\alpha\}} & e^{\{2\alpha\}} - e^{\{-\alpha\}} & e^{\{2\alpha\}} - e^{\{-\alpha\}} \\ e^{\{2\alpha\}} - e^{\{-\alpha\}} & e^{\{2\alpha\}} + 2e^{\{-\alpha\}} & e^{\{2\alpha\}} - e^{\{-\alpha\}} \\ e^{\{2\alpha\}} - e^{\{-\alpha\}} & e^{\{2\alpha\}} - e^{\{-\alpha\}} & e^{\{2\alpha\}} + 2e^{\{-\alpha\}} \end{pmatrix}.
\end{aligned} \tag{161}$$

In this case the operator O_S can be represented in the following more simpler form, *i.e.*

$$O = \begin{pmatrix} c_+ & s_+ & s_+ \\ s_+ & c_+ & s_+ \\ s_+ & s_+ & c_+ \end{pmatrix}, \tag{162}$$

where $c_+ = \frac{1}{3}(e^{\{2\alpha\}} + 2e^{\{-\alpha\}})$, $s_+ = \frac{1}{3}(e^{\{2\alpha\}} - e^{\{-\alpha\}})$ and the cubic equation reduces to the next form:

$$c_+^3 + s_+^3 + t_+^3 - 3c_+s_+t_+ = (c_+ - s_+)^2(c_+ + 2s_+) = 1. \tag{163}$$

Thus, the two parametric ternary $TSO(2)$ group reduces exactly to two known binary symmetries, $\alpha = -\beta$ and $\alpha = \beta$, but for the general case, it produces the new symmetry, in which these two binary symmetry are unified by non-trivial way, (it is not product!)

Let us go further to study so me properties...

$$\begin{aligned}
\tilde{O}_{S1} &= \exp\{j\alpha q_1\} = SS^{-1} \exp\{j\alpha q_1\}S^{-1}S \\
&= S \exp\{j\alpha(S^{-1}q_1S)\} = \exp\{j\alpha q_7\}S^{-1} \\
&= S \left[\sum_{k=0} \frac{(j\alpha)^{3k}}{3k!} + q_7 \sum_{k=0} \frac{(j\alpha)^{3k+1}}{(3k+1)!} + q_7^2 \sum_{k=0} \frac{(j\alpha)^{3k+2}}{(3k+2)!} \right] S^{-1},
\end{aligned} \tag{161}$$

where

$$q_7 = S^{-1}q_1S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix}. \tag{162}$$

Then one can get

$$\begin{aligned}
\tilde{O}_{S1} &= S \left[\begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} + \begin{pmatrix} js & 0 & 0 \\ 0 & j^2s & 0 \\ 0 & 0 & s \end{pmatrix} + \begin{pmatrix} j^2t & 0 & 0 \\ 0 & jt & 0 \\ 0 & 0 & t \end{pmatrix} \right] S^{-1} \\
&= S \begin{pmatrix} c+js+j^2t & 0 & 0 \\ 0 & c+j^2s+jt & 0 \\ 0 & 0 & c+s+t \end{pmatrix} S^{-1} = \begin{pmatrix} c & js & j^2t \\ j^2t & c & js \\ js & j^2t & c \end{pmatrix}
\end{aligned} \tag{161}$$

So we have

$$\tilde{O}_{S1} = \exp\{j\alpha q_1\} = \begin{pmatrix} c & js & j^2t \\ j^2t & c & js \\ js & j^2t & c \end{pmatrix} \tag{162}$$

Similarly,

$$\tilde{\tilde{O}}_{S1} = \exp\{j^2\alpha q_1\} = \begin{pmatrix} c & j^2s & jt \\ j & c & j^2s \\ j^2s & jt & c \end{pmatrix} \tag{163}$$

One can easily to check that

$$\begin{aligned}
O_{S1}\tilde{O}_{S1}\tilde{\tilde{O}}_{S1} &= \exp\{\alpha q_1\} \exp\{\alpha \tilde{q}_1\} \exp\{\alpha \tilde{\tilde{q}}_1\} \\
&= \exp\{\alpha q_1\} \exp\{j\alpha q_1\} \exp\{j^2\alpha q_1\} = (c^3 + s^3 + t^3 - 3cst) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned} \tag{162}$$

One can check that the ternary orthogonal transformations in the following form:

$$O_S = \begin{pmatrix} c & s & t \\ t & c & s \\ s & t & c \end{pmatrix}, \tag{163}$$

with $c^3 + s^3 + t^3 - 3cst = 1$ conserve the cubic forms, *i.e.*

$$(x'_0)^3 + (x'_1)^3 + (x'_2)^3 - 3(x'_0)(x'_1)(x'_2) = x_0^3 + x_1^3 + x_2^3 - 3x_0x_1x_2 \tag{164}$$

or the cubic Laplace equations:

$$\begin{aligned}
\frac{\partial^3 a}{\partial x_0^3} + \frac{\partial^3 a}{\partial x_1^3} + \frac{\partial^3 a}{\partial x_2^3} - 3 \frac{\partial^3 a}{\partial x_0 \partial x_1 \partial x_2} &= 0, \\
\frac{\partial^3 b}{\partial x_0^3} + \frac{\partial^3 b}{\partial x_1^3} + \frac{\partial^3 b}{\partial x_2^3} - 3 \frac{\partial^3 c}{\partial x_0 \partial x_1 \partial x_2} &= 0, \\
\frac{\partial^3 c}{\partial x_0^3} + \frac{\partial^3 c}{\partial x_1^3} + \frac{\partial^3 c}{\partial x_2^3} - 3 \frac{\partial^3 c}{\partial x_0 \partial x_1 \partial x_2} &= 0,
\end{aligned} \tag{162}$$

6 Quaternary C_4 - complex numbers

Consider the quaternary complex numbers

$$z = x_0q_0 + x_1q + x_2q^2 + x_3q^3 \quad (163)$$

where we can consider two cases:

$$A : q^4 = q_0 = 1 \quad (164)$$

or

$$B : q^4 = -q_0 = -1. \quad (165)$$

Let define the conjugation operation of a new complex number:

$$\tilde{q}_0 = q_0 = 1, \quad \tilde{q} = jq, \quad \text{where} \quad j^4 = 1, \quad (166)$$

namely

$$j = \exp i\pi/2. \quad (167)$$

Now one can calculate the norm of this complex number:

$$z\tilde{\tilde{\tilde{z}}} = 1, \quad (168)$$

where

$$\begin{aligned} z &= x_0q_0 + x_1q + x_2q^2 + x_3q^3 \\ \tilde{z} &= x_0q_0 + x_1\tilde{q} + x_2\tilde{q}^2 + x_3\tilde{q}^3 \\ \tilde{\tilde{z}} &= x_0q_0 + x_1\tilde{\tilde{q}} + x_2\tilde{\tilde{q}}^2 + x_3\tilde{\tilde{q}}^3 \\ \tilde{\tilde{\tilde{z}}} &= x_0q_0 + x_1\tilde{\tilde{\tilde{q}}} + x_2\tilde{\tilde{\tilde{q}}}^2 + x_3\tilde{\tilde{\tilde{q}}}^3, \end{aligned} \quad (169)$$

or

$$\begin{aligned} z &= x_0q_0 + x_1q + x_2q^2 + x_3q^3 \\ \tilde{z} &= x_0q_0 + jx_1q + j^2x_2q^2 + j^3x_3q^3 \\ \tilde{\tilde{z}} &= x_0q_0 + j^2x_1q + j^4x_2q^2 + j^6x_3q^3 \\ \tilde{\tilde{\tilde{z}}} &= x_0q_0 + j^3x_1q + j^6x_2q^2 + j^9x_3q^3 \end{aligned} \quad (170)$$

or

$$\begin{aligned}
z &= x_0 q_0 + x_1 q + x_2 q^2 + x_3 q^3 \\
\tilde{z} &= x_0 q_0 + i x_1 q - x_2 q^2 - i x_3 q^3 \\
\tilde{\tilde{z}} &= x_0 q_0 - x_1 q + x_2 q^2 - x_3 q^3 \\
\tilde{\tilde{\tilde{z}}} &= x_0 q_0 - i x_1 q - x_2 q^2 + i x_3 q^3
\end{aligned} \tag{171}$$

We used the following relations:

$\tilde{q} = iq = iq$	$\tilde{\tilde{q}} = i^2 q = -q$	$\tilde{\tilde{\tilde{q}}} = i^3 q = -iq$
$\tilde{q}^2 = i^2 q^2 = -q^2$	$\tilde{\tilde{q}}^2 = i^4 q^2 = q^2$	$\tilde{\tilde{\tilde{q}}}^2 = i^6 q^2 = -q^2$
$\tilde{q}^3 = i^3 q^3 = -iq^3$	$\tilde{\tilde{q}}^3 = i^6 q^3 = -q^3$	$\tilde{\tilde{\tilde{q}}}^3 = i^9 q^3 = -iq^3$

(172)

To find the equation of the surface

$$z \tilde{z} \tilde{\tilde{z}} \tilde{\tilde{\tilde{z}}} = 1, \tag{173}$$

we should take into account the following identities:

$$1 + j + j^2 + j^3 = 0, \quad 1 + j^2 = 0, \quad j + j^3 = 0. \tag{174}$$

In the case A , $q^4 = 1$ the unit quaternary complex numbers determine the following surface:

$$\begin{aligned}
z \tilde{z} \tilde{\tilde{z}} \tilde{\tilde{\tilde{z}}} &= x_0^4 - x_1^4 + x_2^4 - x_3^4 - 2x_0^2 x_2^2 + 2x_1^2 x_3^2 \\
&\quad - 4x_0^2 x_1 x_3 + 4x_1^2 x_0 x_2 - 4x_2^2 x_1 x_3 + 4x_3^2 x_0 x_2 \\
&= [x_0^2 + x_2^2 - 2x_1 x_3]^2 - [x_1^2 + x_3^2 - 2x_0 x_2]^2 \\
&= [x_0^2 + x_1^2 + x_2^2 + x_3^2 - 2x_1 x_3 - 2x_0 x_2][x_0^2 - x_1^2 + x_2^2 - x_3^2 - 2x_1 x_3 + 2x_0 x_2] = \\
&= [(x_0 - x_2)^2 + (x_1 - x_3)^2][(x_0^+ x_2)^2 - (x_1 + x_3)^2] \\
&= (x_0 + x_1 + x_2 + x_3)(x_0 + x_2 - x_1 - x_3)[(x_0 - x_2)^2 + (x_1 - x_3)^2] = 1,
\end{aligned} \tag{170}$$

In the case B , $q^4 = -1$ one can get:

$$\begin{aligned}
z \tilde{z} \tilde{\tilde{z}} \tilde{\tilde{\tilde{z}}} &= x_0^4 + x_1^4 + x_2^4 + x_3^4 + 2x_0^2 x_2^2 + 2x_1^2 x_3^2 \\
&\quad + 4x_0^2 x_1 x_3 - 4x_1^2 x_0 x_2 - 4x_2^2 x_1 x_3 + 4x_3^2 x_0 x_2 \\
&= [x_3^2 - x_1^2 + 2x_0 x_2]^2 + [x_0^2 - x_2^2 + 2x_1 x_3]^2 = 1,
\end{aligned} \tag{169}$$

For illustration consider the Z_4 -holomorphicity for the case A .

Let us consider the function

$$\begin{aligned}
&F(z, \tilde{z}, \tilde{\tilde{z}}, \tilde{\tilde{\tilde{z}}}) \\
&= f_0(x_0, x_1, x_2, x_3) + f_1(x_0, x_1, x_2, x_3)q + f_2(x_0, x_1, x_2, x_3)q^2 + f_3(x_0, x_1, x_2, x_3)q^3
\end{aligned} \tag{168}$$

and her first derivatives:

$$\begin{aligned}
\partial_z F &= \frac{1}{4}\partial_0 F + \frac{1}{4}q^3\partial_1 F + \frac{1}{4}q^2\partial_2 F + \frac{1}{4}q\partial_3 F \\
\partial_{\tilde{z}} F &= \frac{1}{4}\partial_0 F - \frac{i}{4}q^3\partial_1 F - \frac{1}{4}q^2\partial_2 F + \frac{i}{4}q\partial_3 F \\
\partial_{\tilde{\tilde{z}}} F &= \frac{1}{4}\partial_0 F - \frac{1}{4}q^3\partial_1 F + \frac{1}{4}q^2\partial_2 F - \frac{1}{4}q\partial_3 F \\
\partial_{\tilde{\tilde{\tilde{z}}}} F &= \frac{1}{4}\partial_0 F + \frac{i}{4}q^3\partial_1 F - \frac{1}{4}q^2\partial_2 F - \frac{i}{4}q\partial_3 F
\end{aligned} \tag{165}$$

where we used

$$\begin{pmatrix} \partial_z \\ \partial_{\tilde{z}} \\ \partial_{\tilde{\tilde{z}}} \\ \partial_{\tilde{\tilde{\tilde{z}}}} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & q^3 & q^2 & q \\ 1 & -iq^3 & -q^2 & iq \\ 1 & -q^3 & q^2 & -q \\ 1 & iq^3 & -q^2 & -iq \end{pmatrix} \begin{pmatrix} \partial_0 \\ \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \tag{166}$$

where

$$\partial_{z_p} = \frac{\partial}{\partial z_p}, \text{ and } \partial_p = \frac{\partial}{\partial x_p} \quad p = 0, 1, 2, 3, \quad z_1 \equiv \tilde{z}, \quad z_2 \equiv \tilde{\tilde{z}}, \quad z_3 \equiv \tilde{\tilde{\tilde{z}}}.$$

$$\begin{aligned}
\partial_z F &= \frac{1}{4}(\partial_0 f_0 + \partial_1 f_1 + \partial_2 f_2 + \partial_3 f_3) \\
&+ \frac{1}{4}(\partial_0 f_1 + \partial_1 f_2 + \partial_2 f_3 + \partial_3 f_0)q \\
&+ \frac{1}{4}(\partial_0 f_2 + \partial_1 f_3 + \partial_2 f_0 + \partial_3 f_1)q^2 \\
&+ \frac{1}{4}(\partial_0 f_3 + \partial_1 f_0 + \partial_2 f_1 + \partial_3 f_2)q^3
\end{aligned} \tag{163}$$

$$\begin{aligned}
\partial_{z_1} F &= \frac{1}{4}(\partial_0 f_0 - i\partial_1 f_1 - \partial_2 f_2 + i\partial_3 f_3) \\
&+ \frac{1}{4}(\partial_0 f_1 - i\partial_1 f_2 - \partial_2 f_3 + i\partial_3 f_0)q \\
&+ \frac{1}{4}(\partial_0 f_2 - i\partial_1 f_3 - \partial_2 f_0 + i\partial_3 f_1)q^2 \\
&+ \frac{1}{4}(\partial_0 f_3 - i\partial_1 f_0 - \partial_2 f_1 + i\partial_3 f_2)q^3
\end{aligned} \tag{160}$$

$$\begin{aligned}
\partial_{z_2} F &= \frac{1}{4}(\partial_0 f_0 - \partial_1 f_1 + \partial_2 f_2 - \partial_3 f_3) \\
&+ \frac{1}{4}(\partial_0 f_1 - \partial_1 f_2 + \partial_2 f_3 - \partial_3 f_0)q \\
&+ \frac{1}{4}(\partial_0 f_2 - \partial_1 f_3 + \partial_2 f_0 - \partial_3 f_1)q^2 \\
&+ \frac{1}{4}(\partial_0 f_3 - \partial_1 f_0 + \partial_2 f_1 - \partial_3 f_2)q^3
\end{aligned} \tag{157}$$

$$\begin{aligned}
\partial_{z_3} F &= \frac{1}{4}(\partial_0 f_0 + i\partial_1 f_1 - \partial_2 f_2 - i\partial_3 f_3) \\
&+ \frac{1}{4}(\partial_0 f_1 + i\partial_1 f_2 - \partial_2 f_3 - i\partial_3 f_0)q \\
&+ \frac{1}{4}(\partial_0 f_2 + i\partial_1 f_3 - \partial_2 f_0 - i\partial_3 f_1)q^2 \\
&+ \frac{1}{4}(\partial_0 f_3 + i\partial_1 f_0 - \partial_2 f_1 - i\partial_3 f_2)q^3
\end{aligned} \tag{154}$$

In this case we can consider three types of holomoprphicity:

- 1. For the first type of holomorphicity function $F(z_0 z_1, z_2, z_3)$ we have the following three conditions:

$$\frac{\partial F(z, z_1, z_2, z_3)}{\partial z_1} = \frac{\partial F(z, z_1, z_2, z_3)}{\partial z_2} = \frac{\partial F(z, z_1, z_2, z_3)}{\partial z_3} = 0. \tag{155}$$

- 2. For the second type of holomorphicity function $F(z, z_1, z_2, z_3)$ we can take two conditions:

$$\frac{\partial F(z, z_1, z_2, z_3)}{\partial z_2 z} = \frac{\partial F(z, z_1, z_2, z_3)}{\partial z_3} = 0. \tag{156}$$

- 3. For the third type of holomorphicity function $F(z, z_1, z_2, z_3)$ we can take just one condition:

$$\frac{\partial F(z, z_1, z_2, z_3)}{\partial z} = 0. \tag{157}$$

Similarly to the ternary case for $q^3 = 1$, one can get for $q^4 = 1$ for the full Cauchy-Riemann system of the first type:

$$\begin{aligned}
\partial_0 f_0 &= \partial_1 f_1 = \partial_2 f_2 = \partial_3 f_3 \\
\partial_3 f_0 &= \partial_0 f_1 = \partial_1 f_2 = \partial_2 f_3 \\
\partial_2 f_0 &= \partial_3 f_1 = \partial_0 f_2 = \partial_1 f_3 \\
\partial_1 f_0 &= \partial_2 f_1 = \partial_3 f_2 = \partial_0 f_3
\end{aligned} \tag{154}$$

and for quartic Laplace equations one can easily get:

$$\begin{aligned}
&\partial_0^4 f_p - \partial_1^4 f_p + \partial_2^4 f_p - \partial_3^4 f_p - 2\partial_0^2 \partial_2^2 f_p + 2\partial_1^2 \partial_3^2 f_p \\
&- 4\partial_0^2 \partial_1 \partial_3 f_p + 4\partial_1^2 \partial_0 \partial_2 f_p - 4\partial_2^2 \partial_1 \partial_3 f_p + 4\partial_3^2 \partial_0 \partial_2 f_p = 0,
\end{aligned} \tag{153}$$

where $p = 0, 1, 2, 3$. These equations are invariant under $T_4 U(Abel)$ group symmetry, what it follows from $T_4 U(Abel)$ invariance of the quartic form: $zz_1 z_2 z_3$.

Note, that the Cauchy-Riemann conditions can be generalized for any finite C_n group for $q^n = 1$:

We can consider the following four 4×4 matrices from 16 q matrices, which form the quart-quaternion algebra (it will be explained later):

$$q_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, q_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & j^2 \\ j^3 & 0 & 0 & 0 \end{pmatrix}, q_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & j^2 & 0 \\ 0 & 0 & 0 & 1 \\ j^2 & 0 & 0 & 0 \end{pmatrix}, q_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & j^3 & 0 \\ 0 & 0 & 0 & j^2 \\ j & 0 & 0 & 0 \end{pmatrix}, \tag{153}$$

where $j = \exp 2i\pi/4$.

These matrices satisfy to some remarkable relations:

$$\{q_a q_b q_c q_d\}_{S_4} = \eta_{abcd} q_0 \tag{153}$$

with

$$\begin{aligned}
\eta_{1111} &= -\eta_{2222} = \eta_{3333} = -\eta_{4444} = 24 \\
\eta_{1133} &= -\eta_{2244} = 2 \\
-\eta_{1123} &= \eta_{1223} = -\eta_{2334} = \eta_{1344} = 4
\end{aligned} \tag{151}$$

where $j = \exp(\pi/2)$ and q_0 is unit matrix. All others tensor components $\eta_{...}$ are equal zero. Note that the expression $\{q_a q_b q_c q_d\}_{S_4}$ contains the all possible 24 permutations of the S_4 symmetric group, for $a = 1, b = 2, c = 3, d = 4$; 12 for $a = b, c \neq d \neq a$ and etc.

Using these matrices one can get the quaternary Dirac equation:

$$q_1 \frac{\partial \Psi}{\partial x_0} + q_2 \frac{\partial \Psi}{\partial x_1} + q_3 \frac{\partial \Psi}{\partial x_2} + q_4 \frac{\partial \Psi}{\partial x_3} = 0, \quad (152)$$

where

$$\Psi = (\psi_1, \psi_2, \psi_3, \psi_4), \quad (153)$$

is quartet of the wave functions, *i.e.* we introduced the quaternary 1/4 spin structure in \mathbb{R}^4 . The next quaternary structures can appear in $\mathbb{R}^{8,12,\dots}$ spaces.

We can consider the other set of four 4×4 matrices from 16 q matrices, which have the algebraic link to the first set of the four matrices:

$$q_9 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, q_{10} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ j & 0 & 0 & 0 \\ 0 & j^2 & 0 & 0 \\ 0 & 0 & j^3 & 0 \end{pmatrix}, q_{11} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ j^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & j^2 & 0 \end{pmatrix}, q_{12} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ j^3 & 0 & 0 & 0 \\ 0 & j^2 & 0 & 0 \\ 0 & 0 & j & 0 \end{pmatrix}, \quad (153)$$

the second Dirac equation will be:

$$q_9 \frac{\partial \Phi}{\partial x_0} + q_{10} \frac{\partial \Phi}{\partial x_1} + q_{11} \frac{\partial \Phi}{\partial x_2} + q_{12} \frac{\partial \Phi}{\partial x_3} = 0, \quad (154)$$

where

$$\Phi = (\phi_1, \phi_2, \phi_3, \phi_4), \quad (155)$$

(here we are in process...)

In order to diagonalize these equations we must act four times with the same operator and we will get the above mentioned quartic differential equation satisfied by each component ψ_l , $l = 1, 2, 3, 4$.

The quartic Laplace equations should be invariant under Abelian three -parameter group $T_4U(Abel)$:

$$z \rightarrow z' = Uz = \exp\{\phi_1 q + \phi_2 q^2 + \phi_3 q^3\}z = U_1(\phi_1)U_2(\phi_2)U_3(\phi_3) \quad (156)$$

or in the coordinates x_0, x_1, x_2, x_3

$$(x'_0, x'_1, x'_2, x'_3)^t = O \cdot (x_0, x_1, x_2, x_3)^t, \quad (157)$$

where

$$O_A = \begin{pmatrix} m_0 & m_1 & m_2 & m_3 \\ m_3 & m_0 & m_1 & m_2 \\ m_2 & m_3 & m_0 & m_1 \\ m_1 & m_2 & m_3 & m_0 \end{pmatrix}, \quad (158)$$

where

$$\begin{aligned}
 DetO_A = & m_0^4 - m_1^4 + m_2^4 - m_3^4 - 2m_0^2m_2^2 + 2m_1^2m_3^2 \\
 & - 4m_0^2m_1m_3 + 4m_1^2m_0m_2 - 4m_2^2m_1m_3 + 4m_3^2m_0m_2 = 1
 \end{aligned}
 \tag{157}$$

7 \mathbb{C}_6 complex numbers in D=6

Consider the \mathbb{C}_6 complex numbers

$$z = x_0 q_0 + x_1 q + x_2 q^2 + x_3 q^3 + x_4 q^4 + x_5 q^5 \quad (158)$$

where we can consider two cases:

$$A : q^6 = q_0 = 1 \quad (159)$$

and

$$B : q^6 = -q_0 = -1. \quad (160)$$

Let define the conjugation operation of a new complex number:

$$\tilde{q}_0 = q_0 = 1, \quad \tilde{q} = jq, \quad \text{where} \quad j^6 = 1, \quad (161)$$

namely

$$j = \exp i\pi/3. \quad (162)$$

Now one can calculate the norm of this complex number:

$$z \tilde{\tilde{\tilde{\tilde{\tilde{\tilde{z}}}}}} = 1, \quad (163)$$

where

$$\begin{aligned} z &= x_0 q_0 + x_1 q + x_2 q^2 + x_3 q^3 + x_4 q^4 + x_5 q^5 \\ \tilde{z} &= x_0 q_0 + x_1 \tilde{q} + x_2 \tilde{q}^2 + x_3 \tilde{q}^3 + x_4 \tilde{q}^4 + x_5 \tilde{q}^5 \\ \tilde{\tilde{z}} &= x_0 \tilde{\tilde{q}}_0 + x_1 \tilde{\tilde{q}} + x_2 \tilde{\tilde{q}}^2 + x_3 \tilde{\tilde{q}}^3 + x_4 \tilde{\tilde{q}}^4 + x_5 \tilde{\tilde{q}}^5 \\ \tilde{\tilde{\tilde{z}}} &= x_0 \tilde{\tilde{\tilde{q}}}_0 + x_1 \tilde{\tilde{\tilde{q}}} + x_2 \tilde{\tilde{\tilde{q}}}^2 + x_3 \tilde{\tilde{\tilde{q}}}^3 + x_4 \tilde{\tilde{\tilde{q}}}^4 + x_5 \tilde{\tilde{\tilde{q}}}^5, \\ \tilde{\tilde{\tilde{\tilde{z}}}} &= x_0 \tilde{\tilde{\tilde{\tilde{q}}}}_0 + x_1 \tilde{\tilde{\tilde{\tilde{q}}}}} + x_2 \tilde{\tilde{\tilde{\tilde{q}}}}}^2 + x_3 \tilde{\tilde{\tilde{\tilde{q}}}}}^3 + x_4 \tilde{\tilde{\tilde{\tilde{q}}}}}^4 + x_5 \tilde{\tilde{\tilde{\tilde{q}}}}}^5, \\ \tilde{\tilde{\tilde{\tilde{\tilde{z}}}}} &= x_0 \tilde{\tilde{\tilde{\tilde{\tilde{q}}}}} _0 + x_1 \tilde{\tilde{\tilde{\tilde{\tilde{q}}}}} + x_2 \tilde{\tilde{\tilde{\tilde{\tilde{q}}}}}^2 + x_3 \tilde{\tilde{\tilde{\tilde{\tilde{q}}}}}^3 + x_4 \tilde{\tilde{\tilde{\tilde{\tilde{q}}}}}^4 + x_5 \tilde{\tilde{\tilde{\tilde{\tilde{q}}}}}^5, \end{aligned} \quad (158)$$

or

$$\begin{aligned}
z &= x_0q_0 + x_1q + x_2q^2 + x_3q^3 + x_4q^4 + x_5q^5 \\
\tilde{z} &= x_0q_0 + jx_1q + j^2x_2q^2 + j^3x_3q^3 + j^4x_4q^4 + j^5x_5q^5 \\
\tilde{\tilde{z}} &= x_0q_0 + j^2x_1q + j^4x_2q^2 + j^0x_3q^3 + j^2x_4q^4 + j^4x_5q^5 \\
\tilde{\tilde{\tilde{z}}} &= x_0q_0 + j^3x_1q + j^0x_2q^2 + j^3x_3q^3 + j^0x_4q^4 + j^3x_5q^5 \\
\tilde{\tilde{\tilde{\tilde{z}}}} &= x_0q_0 + j^4x_1q + j^2x_2q^2 + j^0x_3q^3 + j^4x_4q^4 + j^2x_5q^5 \\
\tilde{\tilde{\tilde{\tilde{\tilde{z}}}}} &= x_0q_0 + j^5x_1q + j^4x_2q^2 + j^3x_3q^3 + j^2x_4q^4 + jx_5q^5
\end{aligned} \tag{153}$$

We used the following relations:

$$\begin{aligned}
\tilde{q} &= jq, & \tilde{q}^2 &= j^2q^2, & \tilde{q}^3 &= j^3q^3, & \tilde{q}^4 &= j^4q^4, & \tilde{q}^5 &= j^5q^5, \\
\tilde{\tilde{q}} &= j^2q, & \tilde{\tilde{q}}^2 &= j^4q^2, & \tilde{\tilde{q}}^3 &= j^0q^3, & \tilde{\tilde{q}}^4 &= j^2q^4, & \tilde{\tilde{q}}^5 &= j^4q^5, \\
\tilde{\tilde{\tilde{q}}} &= j^3q, & \tilde{\tilde{\tilde{q}}}^2 &= j^0q^2, & \tilde{\tilde{\tilde{q}}}^3 &= j^3q^3, & \tilde{\tilde{\tilde{q}}}^4 &= j^0q^4, & \tilde{\tilde{\tilde{q}}}^5 &= j^3q^5, \\
\tilde{\tilde{\tilde{\tilde{q}}}} &= j^4q, & \tilde{\tilde{\tilde{\tilde{q}}}}^2 &= j^2q^2, & \tilde{\tilde{\tilde{\tilde{q}}}}^3 &= j^0q^3, & \tilde{\tilde{\tilde{\tilde{q}}}}^4 &= j^4q^4, & \tilde{\tilde{\tilde{\tilde{q}}}}^5 &= j^2q^5, \\
\tilde{\tilde{\tilde{\tilde{\tilde{q}}}}} &= j^5q, & \tilde{\tilde{\tilde{\tilde{\tilde{q}}}}}^2 &= j^4q^2, & \tilde{\tilde{\tilde{\tilde{\tilde{q}}}}}^3 &= j^3q^3, & \tilde{\tilde{\tilde{\tilde{\tilde{q}}}}}^4 &= j^2q^4, & \tilde{\tilde{\tilde{\tilde{\tilde{q}}}}}^5 &= jq^5,
\end{aligned} \tag{149}$$

To find the equation of the surface we should take into account the next identities:

$$\begin{aligned}
1 + j + j^2 + j^3 + j^4 + j^5 &= 0 \\
j + j^3 + j^5 &= 0, \quad j - j^2 = 1, \\
1 + j^2 + j^4 &= 0, \quad j^5 - j^4 = 1,
\end{aligned} \tag{147}$$

or

$$\begin{aligned}
j &= \frac{1}{2} + i\frac{\sqrt{3}}{2}, & j^2 &= \frac{-1}{2} + i\frac{\sqrt{3}}{2}, & j^3 &= -1, \\
j^4 &= \frac{-1}{2} - i\frac{\sqrt{3}}{2}, & j^5 &= \frac{1}{2} - i\frac{\sqrt{3}}{2}, & j^6 &= 1.
\end{aligned} \tag{146}$$

For the operations of conjugation one can use the other notations:

$$\begin{aligned}
z^{\{\bar{0}\}} &= x_0q_0+ & x_1q+ & x_2q^2+ & x_3q^3+ & x_4q^4+ & x_5q^5 \\
z^{\{\bar{1}\}} &= x_0q_0+ & jx_1q+ & j^2x_2q^2+ & j^3x_3q^3+ & j^4x_4q^4+ & j^5x_5q^5 \\
z^{\{\bar{2}\}} &= x_0q_0+ & j^2x_1q+ & j^4x_2q^2+ & j^0x_3q^3+ & j^2x_4q^4+ & j^4x_5q^5 \\
z^{\{\bar{3}\}} &= x_0q_0+ & j^3x_1q+ & j^0x_2q^2+ & j^3x_3q^3+ & j^0x_4q^4+ & j^3x_5q^5 \\
z^{\{\bar{4}\}} &= x_0q_0+ & j^4x_1q+ & j^2x_2q^2+ & j^0x_3q^3+ & j^4x_4q^4+ & j^2x_5q^5 \\
z^{\{\bar{5}\}} &= x_0q_0+ & j^5x_1q+ & j^4x_2q^2+ & j^3x_3q^3+ & j^2x_4q^4+ & jx_5q^5
\end{aligned} \tag{147}$$

$$\begin{aligned}
& z[0]^6 - z[1]^6 + z[2]^6 - z[3]^6 + 6z[2]z[3]^4z[4] - \\
& 9z[2]^2z[3]^2z[4]^2 + 2z[2]^3z[4]^3 + z[4]^6 - \\
& 6z[2]^2z[3]^3z[5] + 12z[2]^3z[3]z[4]z[5] - \\
& 6z[3]z[4]^4z[5] - 3z[2]^4z[5]^2 + 9z[3]^2z[4]^2z[5]^2 + \\
& 6z[2]z[4]^3z[5]^2 - 2z[3]^3z[5]^3 - 12z[2]z[3]z[4]z[5]^3 + \\
& 3z[2]^2z[5]^4 - z[5]^6 - \\
& 3z[0]^4(z[3]^2 + 2z[2]z[4] + 2z[1]z[5]) + \\
& 3z[1]^4(z[4]^2 + 2z[3]z[5]) + \\
& 3z[1]^2(3z[2]^2z[3]^2 + 2z[2]^3z[4] - z[4]^4 - \\
& 3z[3]^2z[5]^2 + 6z[2]z[4]z[5]^2) - \\
& 2z[1]^3(z[3]^3 + 6z[2]z[3]z[4] + 3z[2]^2z[5] + z[5]^3) + \\
& 2z[0]^3(z[2]^3 + z[4](3z[1]^2 + z[4]^2 + 6z[3]z[5]) + \\
& 3z[2](2z[1]z[3] + z[5]^2)) - \\
& 6z[1](z[2]^4z[3] - 2z[2]z[3]z[4]^3 + \\
& 3z[2]^2z[4]^2z[5] + (z[4]^2 - z[3]z[5])(z[3]^3 + z[5]^3)) - \\
& 3z[0]^2(2z[1]^3z[3] - z[3]^4 + 6z[1]z[3]z[4]^2 + \\
& 3z[1]^2(z[2]^2 - z[5]^2) + 3z[4]^2(-z[2]^2 + z[5]^2) + \\
& 2z[3](3z[2]^2z[5] + z[5]^3)) + \\
& 6z[0](z[1]^4z[2] + z[2]^3z[3]^2 - z[2]^4z[4] + \\
& 3z[1]^2z[3]^2z[4] - 2z[1]^3z[4]z[5] - \\
& z[2](z[4]^4 - 3z[3]^2z[5]^2) + \\
& z[4](z[3]^2z[4]^2 - 2z[3]^3z[5] + z[5]^4) + \\
& 2z[1](z[2]^3z[5] + z[4]^3z[5] - z[2](z[3]^3 + z[5]^3)))
\end{aligned}$$

(125)

$$\begin{aligned}
& x_0^6 - x_1^6 + 6x_0x_1^4x_2 - 9x_0^2x_1^2x_2^2 + 2x_0^3x_2^3 + x_2^6 - \\
& 6x_0^2x_1^3x_3 + 12x_0^3x_1x_2x_3 - 6x_1x_2^4x_3 - 3x_0^4x_3^2 + \\
& 9x_1^2x_2^2x_3^2 + 6x_0x_2^3x_3^2 - 2x_1^3x_3^3 - 12x_0x_1x_2x_3^3 + \\
& 3x_0^2x_3^4 - x_3^6 + 6x_0^3x_1^2x_4 - 6x_0^4x_2x_4 + \\
& 6x_1^2x_2^3x_4 - 6x_0x_2^4x_4 - 12x_1^3x_2x_3x_4 + \\
& 18x_0x_1^2x_3^2x_4 + 6x_2x_3^4x_4 + 3x_1^4x_4^2 + 9x_0^2x_2^2x_4^2 - \\
& 18x_0^2x_1x_3x_4^2 - 9x_2^2x_3^2x_4^2 - 6x_1x_3^3x_4^2 + \\
& 2x_0^3x_4^3 + 2x_2^3x_4^3 + 12x_1x_2x_3x_4^3 + 6x_0x_3^2x_4^3 - \\
& 3x_1^2x_4^4 - 6x_0x_2x_4^4 + x_4^6 - 6x_0^4x_1x_5 - 6x_1^3x_2^2x_5 + \\
& 12x_0x_1x_2^3x_5 + 6x_1^4x_3x_5 - 18x_0^2x_2^2x_3x_5 - \\
& 6x_2^2x_3^3x_5 + 6x_1x_3^4x_5 - 12x_0x_1^3x_4x_5 + \\
& 12x_0^3x_3x_4x_5 + 12x_2^3x_3x_4x_5 - 12x_0x_3^3x_4x_5 - \\
& 18x_1x_2^2x_4^2x_5 + 12x_0x_1x_4^3x_5 - 6x_3x_4^4x_5 + \\
& 9x_0^2x_1^2x_5^2 + 6x_0^3x_2x_5^2 - 3x_2^4x_5^2 - 9x_1^2x_3^2x_5^2 + \\
& 18x_0x_2x_3^2x_5^2 + 18x_1^2x_2x_4x_5^2 - 9x_0^2x_4^2x_5^2 + \\
& 9x_3^2x_4^2x_5^2 + 6x_2x_4^3x_5^2 - 2x_1^3x_5^3 - 12x_0x_1x_2x_5^3 - \\
& 6x_0^2x_3x_5^3 - 2x_3^3x_5^3 - 12x_2x_3x_4x_5^3 - 6x_1x_4^2x_5^3 + \\
& 3x_2^2x_5^4 + 6x_1x_3x_5^4 + 6x_0x_4x_5^4 - x_5^6
\end{aligned} \tag{108}$$

$$\begin{aligned}
& (x_0 - x_1 + x_2 - x_3 + x_4 - x_5), \\
& (x_0 + x_1 + x_2 + x_3 + x_4 + x_5) \\
& 1/2(2x_0 + x_1 - x_2 - 2x_3 - x_4 + x_5 \\
& -\sqrt{3}\sqrt{-x_1^2 - 2x_1x_2 - x_2^2 + 2x_1x_4 + 2x_2x_4 - x_4^2 + 2x_1x_5 + 2x_2x_5 - 2x_4x_5 - x_5^2}), \\
& 1/2(2x_0 + x_1 - x_2 - 2x_3 - x_4 + x_5 \\
& +\sqrt{3}\sqrt{(-x_1^2 - 2x_1x_2 - x_2^2 + 2x_1x_4 + 2x_2x_4 - x_4^2 + 2x_1x_5 + 2x_2x_5 - 2x_4x_5 - x_5^2)}), \\
& 1/2(2x_0 - x_1 - x_2 + 2x_3 - x_4 - x_5 \\
& -\sqrt{3}\sqrt{(-x_1^2 + 2x_1x_2 - x_2^2 - 2x_1x_4 + 2x_2x_4 - x_4^2 + 2x_1x_5 - 2x_2x_5 + 2x_4x_5 - x_5^2)}), \\
& 1/2(2x_0 - x_1 - x_2 + 2x_3 - x_4 - x_5 \\
& +\sqrt{3}\sqrt{(-x_1^2 + 2x_1x_2 - x_2^2 - 2x_1x_4 + 2x_2x_4 - x_4^2 + 2x_1x_5 - 2x_2x_5 + 2x_4x_5 - x_5^2)})
\end{aligned} \tag{99}$$

In the case A the \mathbb{C}_6 the unit complex numbers define the surface which can be factorized:

$$\begin{aligned}
& (x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6)(x_0 - x_1 + x_2 - x_3 + x_4 - x_5 + x_6) \\
& [(x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - x_0x_2 - x_0x_4 - x_1x_3 - x_1x_5 - x_2x_4 - x_3x_5) \\
+ & (x_0x_1 - 2x_0x_3 + x_0x_5 + x_1x_2 - 2x_1x_4 + x_2x_3 - 2x_2x_5 + x_3x_4 + x_4x_5)] \\
& [(x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - x_0x_2 - x_0x_4 - x_1x_3 - x_1x_5 - x_2x_4 - x_3x_5) \\
- & (x_0x_1 - 2x_0x_3 + x_0x_5 + x_1x_2 - 2x_1x_4 + x_2x_3 - 2x_2x_5 + x_3x_4 + x_4x_5)]
\end{aligned} \tag{95}$$

This surface is invariant under the following transformations:

$$(x'_0, x'_1, x'_2, x'_3, x'_4, x'_5) = O(A)(x_0, x_1, x_2, x_3, x_4, x_5), \tag{96}$$

where

$$O(A) = \begin{pmatrix} m_0 & m_1 & m_2 & m_3 & m_4 & m_5 \\ m_5 & m_0 & m_1 & m_2 & m_3 & m_4 \\ m_4 & m_5 & m_0 & m_1 & m_2 & m_3 \\ m_3 & m_4 & m_5 & m_0 & m_1 & m_2 \\ m_2 & m_3 & m_4 & m_5 & m_0 & m_1 \\ m_1 & m_2 & m_3 & m_4 & m_5 & m_0 \end{pmatrix} \tag{97}$$

where $\text{Det}O(A) = 1$. The expressions for the multi-sin functions one can get through the \mathbb{C}_6 Euler formul:

$$\exp(\phi_1 q + \phi_2 q^2 + \phi_3 q^3 + \phi_4 q^4 \phi_5 q^5) = m_0(\phi_1, \dots, \phi_5)q + \dots + m_5(\phi_0, \dots, \phi_5)q^5. \tag{97}$$

In the case B the \mathbb{C}_6 unit complex numbers define the following surface:

$$\begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 & -x_4 & -x_5 \\ x_5 & x_0 & -x_1 & -x_2 & -x_3 & -x_4 \\ x_4 & x_5 & x_0 & -x_1 & -x_2 & -x_3 \\ x_3 & x_4 & x_5 & x_0 & -x_1 & -x_2 \\ x_2 & x_3 & x_4 & x_5 & x_0 & -x_1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_0 \end{pmatrix} \tag{98}$$

$$\begin{aligned}
& x_0^6 + x_1^6 + 6x_0x_1^4x_2 + 9x_0^2x_1^2x_2^2 + 2x_0^3x_2^3 + x_2^6 + \\
& 6x_0^2x_1^3x_3 + 12x_0^3x_1x_2x_3 - 6x_1x_2^4x_3 + 3x_0^4x_3^2 + \\
& 9x_1^2x_2^2x_3^2 - 6x_0x_2^3x_3^2 - 2x_1^3x_3^3 + 12x_0x_1x_2x_3^3 + \\
& 3x_0^2x_3^4 + x_3^6 + 6x_0^3x_1^2x_4 + 6x_0^4x_2x_4 + \\
& 6x_1^2x_2^3x_4 + 6x_0x_2^4x_4 - 12x_1^3x_2x_3x_4 - \\
& 18x_0x_1^2x_3^2x_4 - 6x_2x_3^4x_4 + 3x_1^4x_4^2 + 9x_0^2x_2^2x_4^2 - \\
& 18x_0^2x_1x_3x_4^2 + 9x_2^2x_3^2x_4^2 + 6x_1x_3^3x_4^2 - \\
& 2x_0^3x_4^3 - 2x_2^3x_4^3 - 12x_1x_2x_3x_4^3 + 6x_0x_3^2x_4^3 + \\
& 3x_1^2x_4^4 - 6x_0x_2x_4^4 + x_4^6 + 6x_0^4x_1x_5 - 6x_1^3x_2^2x_5 - \\
& 12x_0x_1x_2^3x_5 + 6x_1^4x_3x_5 - 18x_0^2x_2^2x_3x_5 + \\
& 6x_2^2x_3^3x_5 - 6x_1x_3^4x_5 + 12x_0x_1^3x_4x_5 - \\
& 12x_0^3x_3x_4x_5 - 12x_2^3x_3x_4x_5 - 12x_0x_3^3x_4x_5 + \\
& 18x_1x_2^2x_4^2x_5 + 12x_0x_1x_4^3x_5 - 6x_3x_4^4x_5 + \\
& 9x_0^2x_1^2x_5^2 - 6x_0^3x_2x_5^2 + 3x_2^4x_5^2 + 9x_1^2x_3^2x_5^2 + \\
& 18x_0x_2x_3^2x_5^2 - 18x_1^2x_2x_4x_5^2 + 9x_0^2x_4^2x_5^2 + \\
& 9x_3^2x_4^2x_5^2 + 6x_2x_4^3x_5^2 + 2x_1^3x_5^3 - 12x_0x_1x_2x_5^3 + \\
& 6x_0^2x_3x_5^3 - 2x_3^3x_5^3 - 12x_2x_3x_4x_5^3 - 6x_1x_4^2x_5^3 + \\
& 3x_2^2x_5^4 + 6x_1x_3x_5^4 - 6x_0x_4x_5^4 + x_5^6
\end{aligned}$$

(81)

$$\begin{aligned}
& z[0]^6 + z[1]^6 + z[2]^6 + z[3]^6 - 6z[2]z[3]^4z[4] + \\
& 9z[2]^2z[3]^2z[4]^2 - 2z[2]^3z[4]^3 + z[4]^6 + \\
& 6z[2]^2z[3]^3z[5] - 12z[2]^3z[3]z[4]z[5] - \\
& 6z[3]z[4]^4z[5] + 3z[2]^4z[5]^2 + 9z[3]^2z[4]^2z[5]^2 + \\
& 6z[2]z[4]^3z[5]^2 - 2z[3]^3z[5]^3 - 12z[2]z[3]z[4]z[5]^3 + \\
& 3z[2]^2z[5]^4 + z[5]^6 + \\
& 3z[0]^4(z[3]^2 + 2z[2]z[4] + 2z[1]z[5]) + \\
& 3z[1]^4(z[4]^2 + 2z[3]z[5]) + \\
& 3z[1]^2(3z[2]^2z[3]^2 + 2z[2]^3z[4] + z[4]^4 + \\
& 3z[3]^2z[5]^2 - 6z[2]z[4]z[5]^2) - \\
& 2z[1]^3(z[3]^3 + 6z[2]z[3]z[4] + 3z[2]^2z[5] - z[5]^3) - \\
& 2z[0]^3(z[2]^3 - z[4](-3z[1]^2 + z[4]^2 + 6z[3]z[5]) + \\
& z[2](6z[1]z[3] - 3z[5]^2)) - \\
& 6z[1](z[2]^4z[3] + 2z[2]z[3]z[4]^3 - \\
& 3z[2]^2z[4]^2z[5] + (-z[4]^2 + z[3]z[5])(z[3]^3 - z[5]^3)) + \\
& 3z[0]^2(2z[1]^3z[3] + z[3]^4 - 6z[1]z[3]z[4]^2 + \\
& 3z[1]^2(z[2]^2 + z[5]^2) + 3z[4]^2(z[2]^2 + z[5]^2) + \\
& z[3](-6z[2]^2z[5] + 2z[5]^3)) - \\
& 6z[0](z[1]^4z[2] - z[2]^3z[3]^2 + z[2]^4z[4] - \\
& 3z[1]^2z[3]^2z[4] + 2z[1]^3z[4]z[5] - \\
& z[2](z[4]^4 - 3z[3]^2z[5]^2) + \\
& z[4](z[3]^2z[4]^2 - 2z[3]^3z[5] - z[5]^4) - \\
& 2z[1](z[2]^3z[5] - z[4]^3z[5] + z[2](-z[3]^3 + z[5]^3)))
\end{aligned}$$

(59)

$$\begin{aligned}
zz^{\{\bar{1}\}}z^{\{\bar{2}\}}z^{\{\bar{3}\}}z^{\{\bar{4}\}}z^{\{\bar{5}\}} &= x_0^6 + x_1^6 + x_2^6 + x_3^6 + x_4^6 + x_5^6 \\
&+ x_0^4x_3^2 + x_1^4x_4^2 + 3x_2^4x_5^2 \\
&+ x_3^4x_0^2 + 3x_4^4x_1^2 + 3x_5^4x_2^2 \\
&+ x_0^4(x_2x_4 + x_1x_5) + x_1^4(x_0x_2 + x_3x_5) \\
&+ x_2^4(x_0x_4 + x_1x_3) + x_3^4(x_1x_5 + x_2x_4) \\
&+ x_4^4(x_0x_2 + x_3x_5) + x_5^4(x_0x_4 + x_1x_3) \\
&+ x_0^3x_2^3 + x_2^3x_4^3 + x_4^3x_0^3 + x_1^3x_3^3 + x_3^3x_5^3 + x_5^3x_1^3 \\
&+ x_0^3(x_1^2x_4 + x_2^2x_5 + x_1x_2x_3 + x_3x_4x_5) \\
&+ x_1^3(x_0^2x_3 + x_2^2x_5 + x_2x_3x_4 + x_0x_4x_5) \\
&+ x_2^3(x_1^2x_4 + x_3^2x_0 + x_0x_1x_5 + x_3x_4x_5) \\
&+ x_3^3(x_2^2x_5 + x_4^2x_1 + x_0x_1x_2 + x_0x_4x_5) \\
&+ x_4^3(x_3^2x_0 + x_5^2x_2 + x_1x_2x_3 + x_0x_1x_5) \\
&+ x_5^3(x_4^2x_1 + x_0^2x_3 + x_0x_1x_2 + x_2x_3x_4) \\
&+ x_0^2x_1^2x_2^2 + x_0^2x_2^2x_4^2 + x_0^2x_1^2x_5^2 + x_0^2x_4^2x_5^2 \\
&+ x_1^2x_2^2x_3^2 + x_1^2x_3^2x_5^2 + x_2^2x_3^2x_4^2 + x_3^2x_4^2x_5^2 \\
&+ x_0^2x_2^2x_3x_5 + x_0^2x_3^2x_1x_5 + x_0^2x_3^2x_2x_4 + x_0^2x_4^2x_1x_3 + x_1^2x_3^2x_0x_4 \\
&+ x_1^2x_5^2x_2x_4 + x_1^2x_4^2x_3x_5 + x_2^2x_4^2x_1x_5 + x_2^2x_5^2x_0x_4 + x_3^2x_5^2x_0x_2 \\
&+ x_0^2x_1x_2x_4x_5 + x_1^2x_0x_2x_3x_5 + x_2^2x_0x_1x_3x_4 \\
&+ x_3^2x_1x_2x_4x_5 + x_4^2x_0x_2x_3x_5 + x_5^2x_0x_1x_3x_4 = 1
\end{aligned} \tag{41}$$

This surface is invariant under transformations:

$$(x'_0, x'_1, x'_2, x'_3, x'_4, x'_5) = O(B)(x_0, x_1, x_2, x_3, x_4, x_5), \tag{42}$$

where

$$O(B) = \begin{pmatrix} m_0 & -m_1 & m_2 & -m_3 & m_4 & -m_5 \\ m_5 & m_0 & -m_1 & m_2 & -m_3 & m_4 \\ -m_4 & m_5 & m_0 & -m_1 & m_2 & -m_3 \\ m_3 & -m_4 & m_5 & m_0 & -m_1 & m_2 \\ -m_2 & m_3 & -m_4 & m_5 & m_0 & -m_1 \\ m_1 & -m_2 & m_3 & -m_4 & m_5 & m_0 \end{pmatrix} \tag{43}$$

where $DetO(B) = 1$.

8 The geometry of ternary generalization of quaternions

Let consider the following construction

$$Q = z_0 + z_1 q_s + z_2 q_s^2, \quad (44)$$

where

$$\begin{aligned} z_0 &= y_0 + qy_1 + q^2 y_2, \\ z_1 &= y_3 + qy_4 + q^2 y_5, \\ z_2 &= y_6 + qx_7 + q^2 y_8 \end{aligned} \quad (42)$$

are the ternary complex numbers and q_s is the new 'imaginary' ternary unit with condition

$$q_s^3 = 1 \quad (43)$$

and

$$q_s q = j q q_s. \quad (44)$$

Then one can see

$$Q = y_0 + qy_1 + q^2 y_2 + y_3 q_s + y_4 q q_s + y_5 q^2 q_s + y_6 q_s^2 + y_7 q q_s^2 + y_8 q^2 q_s^2 \quad (45)$$

We will accept the following notations:

$$\begin{aligned} q &= q_1, & q_s &= q_2, & q^2 q_s^2 &= q_3, & (1) \\ q^2 &= q_4, & q_s^2 &= q_5, & q q_s &= q_6, & (2) \\ q q_s^2 &= q_7, & q^2 q_s &= q_8, & 1 &= q_0. & (0) \end{aligned} \quad (43)$$

Respectively we change the notations of coordinates:

$$\begin{aligned} y_1 &= x_1, & y_3 &= x_2, & y_8 &= x_3, & (1) \\ y_2 &= x_4, & y_6 &= x_5, & y_4 &= x_6, & (2) \\ y_0 &= x_0, & y_7 &= x_7, & y_5 &= x_8, 1 & (0) \end{aligned} \quad (41)$$

In the new notations we have got the following expression:

$$\begin{aligned} Q &= (x_0 + x_7 q_1 q^2 + x_8 q_1^2 q_2) + (x_1 q_1 + x_2 q_2 + x_3 q_1^2 q_2^2) + (x_4 q_1^2 + x_5 q_5^2 + x_6 q_1 q_2) \\ &\equiv z_0(x_0, x_7, x_8) + z_1(x_1, x_2, x_3) + z_2(x_4, x_5, x_6), \end{aligned} \quad (40)$$

where

$$\begin{aligned}
z_0(a, b, c) &= a + bq_1q_2^2 + cq_1^2q_2 \\
z_1(a, b, c) &= aq_1 + bq_2 + cq_1^2q_2^2 \\
z_2(a, b, c) &= aq_1^2 + bq_2^2 + cq_1q_2
\end{aligned} \tag{38}$$

and

$$\{a, b, c\} = \{x_0, x_7, x_8\}, \{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \tag{39}$$

with all possible permutations of triples.

It is easily to check:

—	—	$\{x_0, x_7, x_8\}$	$\{x_1, x_2, x_3\}$	$\{x_4, x_5, x_6\}$
TCl_0	$\begin{matrix} 1 \\ q_1^2q_2 \\ q_1q_2^2 \end{matrix}$	$\begin{matrix} x_0 + x_7q_1q_2^2 + x_8q_1^2q_2 \\ jx_0q_1q_2^2 + j^2x_7q_1^2q_2 + x_8 \\ jx_0q_1^2q_2 + x_7 + j^2x_8q_1q_2^2 \end{matrix}$	$\begin{matrix} x_1q_1 + x_2q_2 + x_3q_1^2q_2^2 \\ j^2x_1q_1^2q_2^2 + jx_2q_1 + x_3q_2 \\ x_1q_2 + jx_2q_1^2q_2^2 + j^2x_3q_1 \end{matrix}$	$\begin{matrix} x_4q_1^2 + x_5q_2^2 + x_6q_1q_2 \\ x_4q_2^2 + jx_5q_1q_2 + j^2x_6q_1^2 \\ j^2x_4q_1q_2 + jx_5q_1^2 + x_6q_2^2 \end{matrix}$
TCl_1	$\begin{matrix} q_1 \\ q_2 \\ q_1^2q_2^2 \end{matrix}$	$\begin{matrix} x_0q_1^2 + x_7q_2^2 + x_8q_1q_2 \\ x_0q_2^2 + jx_7q_1q_2 + j^2x_8q_1^2 \\ j^2x_0q_1q_2 + jx_7q_1^2 + x_8q_2^2 \end{matrix}$	$\begin{matrix} x_1 + x_2q_1^2q_2 + x_3q_1q_2^2 \\ jx_1q_1q_2^2 + x_2 + j^2x_3q_1^2q_2 \\ jx_1q_1^2q_2 + j^2x_2q_1q_2^2 + x_3 \end{matrix}$	$\begin{matrix} x_4q_1 + x_5q_1^2q_2^2 + x_6q_2 \\ j^2x_4q_1^2q_2^2 + x_5q_2 + jx_6q_1 \\ x_4q_2 + j^2x_5q_1 + jx_6q_1^2q_2^2 \end{matrix}$
TCl_2	$\begin{matrix} q_1^2 \\ q_2^2 \\ q_1q_2 \end{matrix}$	$\begin{matrix} x_0q_1 + x_7q_1^2q_2^2 + x_8q_2 \\ x_0q_2 + j^2x_7q_1 + jx_8q_1^2q_2^2 \\ j^2x_0q_1^2q_2^2 + x_7q_2 + jx_8q_1 \end{matrix}$	$\begin{matrix} x_1q_1^2 + x_2q_1q_2 + x_3q_2^2 \\ j^2x_1q_1q_2 + x_2q_2^2 + jx_3q_1^2 \\ x_1q_2^2 + j^2x_2q_1^2 + jx_3q_1q_2 \end{matrix}$	$\begin{matrix} x_4 + x_5q_1q_2^2 + x_6q_1^2q_2 \\ jx_4q_1^2q_2 + x_5 + j^2x_6q_1q_2^2 \\ jx_4q_1q_2^2 + j^2x_5q_1^2q_2 + x_6 \end{matrix}$

$$\begin{aligned}
Q &= [z_0(x_0, x_7, x_8) + z_1(x_1, x_2, x_3) + z_2(x_4, x_5, x_6)] \\
&= q_1q_2^2[z_0(x_7, x_8, x_0) + z_1(x_3, x_1, x_2) + z_2(x_5, x_6, x_4)] \\
&= q_1^2q_2[z_0(x_8, x_0, x_7) + z_1(x_2, x_3, x_1) + z_2(x_6, x_4, x_5)]
\end{aligned} \tag{38}$$

$$\begin{aligned}
Q &= q_1[z_0(x_1, x_3, x_2) + z_1(x_4, x_6, x_5) + z_2(x_0, x_7, x_8)] \\
&= q_2[z_0(x_2, x_3, x_1) + z_1(x_5, x_4, x_6) + z_2(x_0, x_8, x_7)] \\
&= q_1^2q_2^2[z_0(x_3, x_2, x_1) + z_1(x_6, x_4, x_5) + z_2(x_0, x_7, x_8)]
\end{aligned} \tag{36}$$

$$\begin{aligned}
Q &= q_1^2[z_0(x_4, x_5, x_6) + z_1(x_0, x_8, x_7) + z_2(x_1, x_3, x_2)] \\
&= q_2^2[z_0(x_5, x_6, x_4) + z_1(x_7, x_0, x_8) + z_2(x_3, x_2, x_1)] \\
&= q_1q_2[z_0(x_6, x_4, x_5) + z_1(x_8, x_7, x_0) + z_2(x_2, x_1, x_3)]
\end{aligned} \tag{34}$$

Now we can rewrite the expression for Q , \tilde{Q} , and $\tilde{\tilde{Q}}$ in the following way:

$$\begin{aligned} Q &= z_2(x_4, x_5, x_6) + q_1 z_2(x_0, x_7, x_8) + q_1^2 z_2(x_1, x_3, x_2) \\ \tilde{Q} &= \tilde{z}_2(x_4, x_5, x_6) + j q_1 \tilde{z}_2(x_0, x_7, x_8) + j^2 q_1^2 \tilde{z}_2(x_1, x_3, x_2) \\ \tilde{\tilde{Q}} &= \tilde{\tilde{z}}_2(x_4, x_5, x_6) + j^2 q_1 \tilde{\tilde{z}}_2(x_0, x_7, x_8) + j q_1^2 \tilde{\tilde{z}}_2(x_1, x_3, x_2) \end{aligned} \quad (32)$$

where we accept that

$$\begin{aligned} \tilde{q}_1 &= j q_1, & \tilde{q}_2 &= j q_2, \\ \tilde{\tilde{q}}_1 &= j^2 q_1, & \tilde{\tilde{q}}_2 &= j^2 q_2. \end{aligned} \quad (31)$$

We would like to calculate the product $Q\tilde{Q}\tilde{\tilde{Q}}$ what in general contains itself $9 \times 9 \times 9 = 729$ terms, *i.e.*

$$Q \times \tilde{Q} \times \tilde{\tilde{Q}} = A_0(x_0, \dots, x_8)q_0 + A_1(x_0, \dots, x_8)q_1 + \dots + A_8(x_0, \dots, x_8)q_8 \quad (31)$$

In general in this product one can meet inside A_p ($p = 0, 1, \dots, 8$) the following term structures:

$$\begin{aligned} x_p^3, & \quad p = 0, 1, \dots, 8 \\ x_p^2 x_k, & \quad p \neq r \\ x_p x_k x_l, & \quad p \neq k \neq l. \end{aligned} \quad (29)$$

For this expansion one can easily see that

$$729 = 9(x_p^3 - \text{terms}) + 72 \times 3(x_p^2 x_k - \text{terms}) + 84 \times 6(x_p x_k x_l - \text{terms}). \quad (30)$$

Note that we would like to save just terms proportional to $q_0 = 1$ - terms, *i.e.* to find the A_0 magnitude. All others must be equal to zero. Since $q_p^3 = 1$ for $p = 0, 1, \dots, 8$ A_0 contains the first nine pure cubic terms.

We can see how in the product $Q\tilde{Q}\tilde{\tilde{Q}}$ vanish the terms $72 \times 3(x_p^2 x_k - \text{terms})$. From the expression

respectively. Now one can see that all terms disappear. In this product we can find the nonvanishing terms proportional $q_0 = 1$:

$$\begin{aligned} x_0^3 + x_7^3 + x_8^3 - 3x_0 x_7 x_8, \\ x_1^3 + x_2^3 + x_3^3 - 3x_1 x_2 x_3, \\ x_4^3 + x_5^3 + x_6^3 - 3x_4 x_5 x_6, \end{aligned} \quad (28)$$

where we took into account that

$$\begin{aligned} q_0 q_7 q_8 &\sim 1 \\ q_1 q_2 q_3 &\sim 1 \\ q_4 q_5 q_6 &\sim 1. \end{aligned} \tag{26}$$

Also, one can also find the other combination proportional to 1:

$$\begin{aligned} q_0(q_1 q_4 + q_2 q_5 + q_3 q_6) &\sim 1 \\ q_7(q_1 q_5 + q_2 q_6 + q_3 q_1) &\sim 1 \\ q_8(q_1 q_6 + q_2 q_4 + q_3 q_5) &\sim 1. \end{aligned} \tag{24}$$

So, we have got in the triple product the

$$9(x_p^3) + 12 \times 6(x_p x_k x_l) = 81(\text{terms}). \tag{25}$$

The $729 - 81 = 72 \times 3 + 72 \times 6 = 648$ terms are vanished.

Thus, we expect to get the equation for the unit ternary “quaternion” surface in the following form:

$$\begin{aligned} &x_0^3 + x_7^3 + x_8^3 - 3x_0 x_7 x_8 \\ &x_1^3 + x_2^3 + x_3^3 - 3x_1 x_2 x_3 \\ &x_0^4 + x_5^3 + x_6^3 - 3x_4 x_5 x_6 \\ &- 3x_0(x_1 x_4 + x_2 x_5 + x_3 x_6) \\ &- 3x_7(x_1 x_5 + x_2 x_6 + x_3 x_4) \\ &- 3x_8(x_1 x_6 + x_2 x_4 + x_3 x_5) = 1 \end{aligned} \tag{20}$$

In this product we can find the nonvanishing terms proportional $q_0 = 1$:

$$\begin{aligned} &x_0^3 + x_7^3 + x_8^3 - 3x_0 x_7 x_8, \\ &x_1^3 + x_2^3 + x_3^3 - 3x_1 x_2 x_3, \\ &x_0^4 + x_5^3 + x_6^3 - 3x_4 x_5 x_6, \end{aligned} \tag{18}$$

where we took into account that

$$\begin{aligned} q_0 q_7 q_8 &\sim 1 \\ q_1 q_2 q_3 &\sim 1 \\ q_4 q_5 q_6 &\sim 1. \end{aligned} \tag{16}$$

Also, one can also find the other combination proportional to 1:

$$\begin{aligned}
q_0(q_1q_4 + q_2q_5 + q_3q_6) &\sim 1 \\
q_7(q_1q_5 + q_2q_6 + q_3q_1) &\sim 1 \\
q_8(q_1q_6 + q_2q_4 + q_3q_5) &\sim 1.
\end{aligned} \tag{14}$$

$$\begin{aligned}
Q &= (x_0 + x_7q_7 + x_8q_8) + (x_1q_1 + x_2q_2 + x_3q_3) + (x_4q_4 + x_5q_5 + x_6q_6) \\
\tilde{Q} &= (x_0 + jx_7q_7 + j^2x_8q_8) + j(x_1q_1 + x_2q_2 + x_3q_3) + j^2(x_4q_4 + x_5q_5 + x_6q_6) \\
\tilde{\tilde{Q}} &= (x_0 + j^2x_7q_7 + jx_8q_8) + j^2(x_1q_1 + x_2q_2 + x_3q_3) + j(x_4q_4 + x_5q_5 + x_6q_6)
\end{aligned} \tag{12}$$

$$Q_1 = q \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{13}$$

$$Q_2 = q^2 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix}, \tag{14}$$

$$Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j^2 \\ j & 0 & 0 \end{pmatrix}, \tag{14}$$

$$Q_4 = Q_1^2 = q^2 \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{15}$$

$$Q_5 = Q_2^2 = q \begin{pmatrix} 0 & 0 & j \\ 1 & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}, \tag{16}$$

$$Q_6 = Q_3^2 = \begin{pmatrix} 0 & 0 & j^2 \\ 1 & 0 & 0 \\ 0 & j & 0 \end{pmatrix}. \tag{17}$$

$$Q_7 = \begin{pmatrix} j & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (18)$$

$$Q_8 = \begin{pmatrix} j^2 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (19)$$

$$Q_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (20)$$

$$\begin{aligned} Q_1 Q_2 &= j^2 Q_6, \quad Q_2 Q_3 = j^2 q^2 Q_4, \quad Q_3 Q_1 = j^2 q Q_5 \\ Q_2 Q_1 &= j Q_6, \quad Q_3 Q_2 = j q^2 Q_4, \quad Q_1 Q_3 = j q Q_5 \end{aligned} \quad (19)$$

$$\begin{aligned} Q_4 Q_5 &= j^2 Q_3, \quad Q_5 Q_6 = j^2 q Q_1, \quad Q_6 Q_4 = j^2 q^2 Q_2 \\ Q_5 Q_4 &= j Q_3, \quad Q_6 Q_5 = j q Q_1, \quad Q_4 Q_6 = j q^2 Q_2 \end{aligned} \quad (18)$$

$$Q_1 Q_5 = q^2 j \begin{pmatrix} j^2 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q_5 Q_1 = q^2 j^2 \begin{pmatrix} j^2 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (19)$$

$$Q_1 Q_6 = q j^2 \begin{pmatrix} j & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q_6 Q_1 = q j \begin{pmatrix} j & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (20)$$

$$Q_2 Q_4 = q j^2 \begin{pmatrix} j & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q_4 Q_2 = q j \begin{pmatrix} j & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (21)$$

$$Q_2 Q_6 = q^2 j \begin{pmatrix} j^2 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q_6 Q_2 = q^2 j^2 \begin{pmatrix} j^2 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (22)$$

$$Q_3Q_4 = q^2j \begin{pmatrix} j^2 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q_4Q_3 = q^2j^2 \begin{pmatrix} j^2 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (23)$$

$$Q_3Q_5 = qj^2 \begin{pmatrix} j & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q_5Q_3 = qj \begin{pmatrix} j & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (24)$$

$$\begin{aligned} Q_1Q_4 &= Q_0, \quad Q_1Q_5 = q^2jQ_8, \quad Q_1Q_6 = qj^2Q_7 \\ Q_4Q_1 &= Q_0, \quad Q_5Q_1 = q^2j^2Q_8, \quad Q_6Q_1 = jqQ_7 \end{aligned} \quad (23)$$

$$\begin{aligned} Q_2Q_4 &= qj^2Q_7, \quad Q_2Q_5 = Q_0, \quad Q_2Q_6 = j^2q^2Q_7 \\ Q_4Q_2 &= qjQ_7, \quad Q_5Q_2 = Q_0, \quad Q_6Q_2 = jq^2Q_7 \end{aligned} \quad (22)$$

$$\begin{aligned} Q_3Q_4 &= q^2jQ_8, \quad Q_3Q_5 = qj^2Q_7, \quad Q_3Q_6 = Q_0 \\ Q_4Q_3 &= q^2j^2jQ_8, \quad Q_5Q_3 = qjQ_7, \quad Q_6Q_3 = Q_0 \end{aligned} \quad (21)$$

The ternary conjugation include two operations:

- 1. $\tilde{q} = jq$;
- 2. $\{1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 1\}$.

Let us check the second operation. For this consider two 3×3 matrices:

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ c_2 & a_2 & b_2 \\ b_3 & c_3 & a_3 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} u_1 & v_1 & w_1 \\ w_2 & u_2 & v_2 \\ v_3 & w_3 & u_3 \end{pmatrix} \quad (22)$$

Then

$$\tilde{A} = \begin{pmatrix} a_3 & b_3 & c_3 \\ c_1 & a_1 & b_1 \\ b_2 & c_2 & a_2 \end{pmatrix}, \quad \text{and} \quad \tilde{B} = \begin{pmatrix} u_3 & v_3 & w_3 \\ w_1 & u_1 & v_1 \\ v_2 & w_2 & u_2 \end{pmatrix} \quad (23)$$

respectively.

Take the product of these two matrices in both cases:

$$C = A \cdot B = \begin{pmatrix} a_1u_1 + b_1w_2 + c_1v_3 & a_1v_1 + b_1u_2 + c_1w_3 & a_1w_1 + b_1v_2 + c_1u_3 \\ c_2u_1 + a_2w_2 + b_2v_3 & c_2v_1 + a_2u_2 + b_2w_3 & c_2w_1 + a_2v_2 + b_2u_3 \\ b_3u_1 + c_3w_2 + a_3v_3 & b_3v_1 + c_3u_2 + a_3w_3 & b_3w_1 + c_3v_2 + a_3u_3 \end{pmatrix}, \quad (24)$$

$$\tilde{A} \cdot \tilde{B} = \begin{pmatrix} b_3w_1 + c_3v_2 + a_3u_3 & b_3u_1 + c_3w_2 + a_3v_3 & b_3v_1 + c_3u_2 + a_3w_3 \\ a_1w_1 + b_1v_2 + c_1u_3 & a_1u_1 + b_1w_2 + c_1v_3 & a_1v_1 + b_1u_2 + c_1w_3 \\ c_2w_1 + a_2v_2 + b_2u_3 & c_2u_1 + a_2w_2 + b_2v_3 & c_2v_1 + a_2u_2 + b_2w_3 \end{pmatrix}, \quad (25)$$

Compare the last expression with the expression of \tilde{C} one can see that:

$$(A \tilde{\cdot} B) = \tilde{A} \cdot \tilde{B}. \quad (26)$$

$$\tilde{Q}_1 = jq \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = jQ_1, \quad (27)$$

$$\tilde{Q}_2 = j^2q^2 \begin{pmatrix} 0 & j^2 & 0 \\ 0 & 0 & 1 \\ j & 0 & 0 \end{pmatrix} = jQ_2, \quad (28)$$

$$\tilde{Q}_3 = \begin{pmatrix} 0 & j & 0 \\ 0 & 0 & 1 \\ j^2 & 0 & 0 \end{pmatrix} = jQ_3, \quad (29)$$

$$\tilde{Q}_4 = j^2q^2 \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = j^2Q_4, \quad (30)$$

$$\tilde{Q}_5 = jq \begin{pmatrix} 0 & 0 & j^2 \\ j & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = j^2Q_5, \quad (31)$$

$$\tilde{Q}_6 = \begin{pmatrix} 0 & 0 & j \\ j^2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = j^2Q_6 \quad (31)$$

$$Q_7 = j \begin{pmatrix} j & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (32)$$

$$Q_8 = j^2 \begin{pmatrix} j^2 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (33)$$

$$\begin{aligned} Q_7 Q_1 &= q Q_2, & Q_7 Q_2 &= q Q_3, & Q_7 Q_3 &= q Q_1 \\ Q_8 Q_2 &= q^2 Q_1, & Q_8 Q_3 &= q^2 Q_2, & Q_8 Q_1 &= q^2 Q_3. \end{aligned} \quad (32)$$

$$\begin{aligned} Q_7 Q_4 &= q Q_6, & Q_7 Q_6 &= q Q_5, & Q_7 Q_5 &= q Q_4 \\ Q_8 Q_4 &= q^2 Q_5, & Q_8 Q_5 &= q^2 Q_6, & Q_8 Q_6 &= q^2 Q_4. \end{aligned} \quad (31)$$

$$q_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (31)$$

9 Ternary TU(3)-algebra

We can consider the 3×3 matrix realization of q - algebra:

$$\begin{aligned} q_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, q_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix}, q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j^2 \\ j & 0 & 0 \end{pmatrix}, \\ q_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, q_5 = \begin{pmatrix} 0 & 0 & j \\ 1 & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}, q_6 = \begin{pmatrix} 0 & 0 & j^2 \\ 1 & 0 & 0 \\ 0 & j & 0 \end{pmatrix}, \\ q_7 &= j \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix}, q_8 = j^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & j \end{pmatrix}, q_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \quad (29)$$

which satisfy to the ternary algebra:

$$\{A, B, C\}_{S_3} = ABC + BCA + CAB - BAC - ACB - CBA. \quad (30)$$

Here $j = \exp(2i\pi/3)$ and S_3 is the permutation group of three elements.

On the next table we can give the ternary commutation relations for the matrices q_k :

$$\{q_k, q_l, q_m\}_{S_3} = f_{klm}^n q_n. \quad (30)$$

One can check that each triple commutator $\{q_k, q_l, q_m\}$, defined by triple numbers, $\{klm\}$ with $k, l, m = 0, 1, 2, \dots, 8$, gives just one matrix q_n with the corresponding coefficient f_{klm}^n giving in the table:

The q_k elements satisfy to the ternary algebra:

$$\{A, B, C\}_{S_3} = ABC + BCA + CAB - BAC - ACB - CBA. \quad (31)$$

Here $j = \exp(2i\pi/3)$ and S_3 is the permutation group of three elements.

On the next table we can give the ternary commutation relations for the matrices q_k :

$$\{q_k, q_l, q_m\}_{S_3} = f_{klm}^n q_n. \quad (31)$$

One can check that each triple commutator $\{q_k, q_l, q_m\}$, defined by triple numbers, $\{klm\}$ with $k, l, m = 0, 1, 2, \dots, 8$, gives just one matrix q_n with the corresponding coefficient f_{klm}^n giving in the table:

One can find $C_9^2 = 84$ ternary commutation relations. But there one can see that there are also $C_8^2 = 28$ commutation relations which correspond to the $su(3)$ algebra! Therefore, it is naturally to represent the q -numbers as ternary generalization of quaternions. If one can take from S_3 commutation relations $C = q_0$ the commutation relations naturally are going to S_2 Lie commutation relations:

Table 3: *The ternary commutation relations*

N	$\{klm\} \rightarrow \{n\}$	f_{klm}^n	N	$\{klm\} \rightarrow \{n\}$	f_{klm}^n	N	$\{klm\} \rightarrow \{n\}$	f_{klm}^n
1	$\{123\} \rightarrow \{0\}$	$3(j^2 - j)$	2	$\{124\} \rightarrow \{2\}$	$j(1 - j)$	3	$\{125\} \rightarrow \{1\}$	$2(j^2 - j)$
4	$\{126\} \rightarrow \{3\}$	$j(1 - j)$	5	$\{127\} \rightarrow \{5\}$	$2(1 - j)$	6	$\{128\} \rightarrow \{4\}$	$2(j^2 - 1)$
7	$\{120\} \rightarrow \{6\}$	$(j^2 - j)$	8	$\{134\} \rightarrow \{3\}$	$(j^2 - j)$	9	$\{135\} \rightarrow \{2\}$	$2(j - j^2)$
10	$\{136\} \rightarrow \{1\}$	$(j^2 - j)$	11	$\{137\} \rightarrow \{4\}$	$2(j - 1)$	12	$\{138\} \rightarrow \{6\}$	$2(1 - j^2)$
13	$\{130\} \rightarrow \{5\}$	$(j - j^2)$	14	$\{145\} \rightarrow \{5\}$	$(j - j^2)$	15	$\{146\} \rightarrow \{6\}$	$(j^2 - j)$
16	$\{147\} \rightarrow \{7\}$	$(j^2 - j)$	17	$\{148\} \rightarrow \{8\}$	$(j - j^2)$	18	$\{140\} \rightarrow \bar{O}$	0
19	$\{156\} \rightarrow \{4\}$	$2j(j - 1)$	20	$\{157\} \rightarrow \{0\}$	$3(1 - j)$	21	$\{158\} \rightarrow \{7\}$	$2(1 - j)$
22	$\{150\} \rightarrow \{8\}$	$(1 - j)$	23	$\{167\} \rightarrow \{8\}$	$2(1 - j^2)$	24	$\{168\} \rightarrow \{0\}$	$3(1 - j^2)$
25	$\{160\} \rightarrow \{7\}$	$(1 - j^2)$	26	$\{178\} \rightarrow \{1\}$	$(j - j^2)$	27	$\{170\} \rightarrow \{2\}$	$(j - 1)$
28	$\{180\} \rightarrow \{3\}$	$(j^2 - 1)$	29	$\{234\} \rightarrow \{1\}$	$2(j^2 - j)$	30	$\{235\} \rightarrow \{3\}$	$(j - j^2)$
31	$\{236\} \rightarrow \{2\}$	$(j - j^2)$	32	$\{237\} \rightarrow \{6\}$	$2(1 - j)$	33	$\{238\} \rightarrow \{5\}$	$2(j^2 - 1)$
34	$\{230\} \rightarrow \{4\}$	$(j^2 - j)$	35	$\{245\} \rightarrow \{4\}$	$(j - j^2)$	36	$\{246\} \rightarrow \{5\}$	$2(j - j^2)$
37	$\{247\} \rightarrow \{8\}$	$2(1 - j^2)$	38	$\{248\} \rightarrow \{0\}$	$3(1 - j^2)$	39	$\{240\} \rightarrow \{7\}$	$(1 - j^2)$
40	$\{256\} \rightarrow \{6\}$	$(j - j^2)$	41	$\{257\} \rightarrow \{7\}$	$(j^2 - j)$	42	$\{258\} \rightarrow \{8\}$	$(j - j^2)$
43	$\{250\} \rightarrow \bar{O}$	0	44	$\{267\} \rightarrow \{0\}$	$3(1 - j)$	45	$\{268\} \rightarrow \{7\}$	$2(1 - j)$
46	$\{260\} \rightarrow \{8\}$	$(1 - j)$	47	$\{278\} \rightarrow \{2\}$	$(j - j^2)$	48	$\{270\} \rightarrow \{3\}$	$(j - 1)$
49	$\{280\} \rightarrow \{1\}$	$(j^2 - 1)$	50	$\{345\} \rightarrow \{6\}$	$2(j^2 - j)$	51	$\{346\} \rightarrow \{4\}$	$(j^2 - j)$
52	$\{347\} \rightarrow \{0\}$	$3(1 - j)$	53	$\{348\} \rightarrow \{7\}$	$2(1 - j)$	54	$\{340\} \rightarrow \{8\}$	$(1 - j)$
55	$\{356\} \rightarrow \{5\}$	$j - j^2$	56	$\{357\} \rightarrow \{8\}$	$2(1 - j^2)$	57	$\{358\} \rightarrow \{0\}$	$3(1 - j^2)$
58	$\{350\} \rightarrow \{7\}$	$(1 - j^2)$	59	$\{367\} \rightarrow \{7\}$	$(j^2 - j)$	60	$\{368\} \rightarrow \{8\}$	$(j - j^2)$
61	$\{360\} \rightarrow \bar{O}$	0	62	$\{378\} \rightarrow \{3\}$	$(j - j^2)$	63	$\{370\} \rightarrow \{1\}$	$(j - 1)$
64	$\{380\} \rightarrow \{2\}$	$(j^2 - 1)$	65	$\{456\} \rightarrow \{0\}$	$3(j^2 - j)$	66	$\{457\} \rightarrow \{1\}$	$2(1 - j)$
67	$\{458\} \rightarrow \{2\}$	$2(j^2 - 1)$	68	$\{450\} \rightarrow \{3\}$	$(j^2 - j)$	69	$\{467\} \rightarrow \{1\}$	$2(j - 1)$
70	$\{468\} \rightarrow \{1\}$	$2(1 - j^2)$	71	$\{460\} \rightarrow \{2\}$	$(j - j^2)$	72	$\{478\} \rightarrow \{4\}$	$(j^2 - j)$
73	$\{470\} \rightarrow \{6\}$	$(1 - j)$	74	$\{480\} \rightarrow \{5\}$	$(1 - j^2)$	75	$\{567\} \rightarrow \{2\}$	$2(1 - j)$
76	$\{568\} \rightarrow \{3\}$	$2(j^2 - j)$	77	$\{560\} \rightarrow \{1\}$	$(j^2 - j)$	78	$\{578\} \rightarrow \{5\}$	$(j^2 - j)$
79	$\{570\} \rightarrow \{4\}$	$(1 - j)$	80	$\{580\} \rightarrow \{6\}$	$(1 - j^2)$	81	$\{678\} \rightarrow \{6\}$	$(j^2 - j)$
82	$\{670\} \rightarrow \{5\}$	$(1 - j)$	83	$\{680\} \rightarrow \{4\}$	$(1 - j^2)$	84	$\{780\} \rightarrow \bar{O}$	0

$$\{q_a, q_b, q_0\}_{S_3} = q_a q_b q_0 + q_b q_0 q_a + q_0 q_a q_b - q_b q_a q_0 - q_a q_0 q_b - q_0 q_b q_a = q_a q_b - q_b q_a, \quad (31)$$

where $a \neq b \neq 0$. On the table such 28- cases one can see $\{kl0\}$.

We can consider the 3×3 matrix realization of q - algebra:

$$\begin{aligned}
q_1 &= q \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, q_2 = q^2 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix}, q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j^2 \\ j & 0 & 0 \end{pmatrix}, \\
q_4 &= q^2 \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, q_5 = q \begin{pmatrix} 0 & 0 & j \\ 1 & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}, q_6 = \begin{pmatrix} 0 & 0 & j^2 \\ 1 & 0 & 0 \\ 0 & j & 0 \end{pmatrix}, \\
q_7 &= q^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix}, q_8 = q \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & j \end{pmatrix}, q_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned} \tag{29}$$

which satisfy to the ternary algebra:

$$\{A, B, C\}_{S_3} = ABC + BCA + CAB - BAC - ACB - CBA. \tag{30}$$

Here $j = \exp(2i\pi/3)$ and S_3 is the permutation group of three elements.

On the next table we can give the ternary commutation relations for the matrices q_k :

$$\{q_k, q_l, q_m\}_{S_3} = f_{klm}^n q_n. \tag{30}$$

One can check that each triple commutator $\{q_k, q_l, q_m\}$, defined by triple numbers, $\{klm\}$ with $k, l, m = 0, 1, 2, \dots, 8$, gives just one matrix q_n with the corresponding coefficient f_{klm}^n giving in the table:

10 The geometrical representations of ternary "quaternions"

Let us define the following product:

$$\begin{aligned}
\hat{Q} = \sum_{a=0}^{a=8} \{x_a q_a\} &= \begin{pmatrix} x_0 + jx_7 + j^2x_8 & x_1 + x_2 + x_3 & x_4 + jx_5 + j^2x_6 \\ x_4 + x_5 + x_6 & x_0 + j^2x_7 + jx_8 & x_1 + jx_2 + j^2x_3 \\ x_1 + j^2x_2 + jx_3 & x_4 + j^2x_5 + jx_6 & x_0 + x_7 + x_8 \end{pmatrix} \\
&= \begin{pmatrix} \tilde{z}_0 & z_1 & \tilde{z}_2 \\ z_2 & \tilde{\tilde{z}}_0 & \tilde{z}_1 \\ \tilde{\tilde{z}}_1 & \tilde{\tilde{z}}_2 & z_0 \end{pmatrix}.
\end{aligned} \tag{30}$$

$$\begin{aligned}
& (z_0 z_1 z_2 + \tilde{z}_0 \tilde{z}_1 \tilde{z}_2 + \tilde{\tilde{z}}_0 \tilde{\tilde{z}}_1 \tilde{\tilde{z}}_2) \\
= & [(x_0 + x_7 + x_8)(x_1 + x_2 + x_3)(x_4 + x_5 + x_6)] \\
+ & [(x_0 + jx_7 + j^2 x_8)(x_1 + jx_2 + j^2 x_3)(x_4 + j^2 x_5 + jx_6)] \\
+ & [(x_0 + j^2 x_7 + jx_8)(x_1 + j^2 x_2 + jx_3)(x_4 + jx_5 + j^2 x_6)] \\
= & [(x_0 + x_7 + x_8) \\
& \cdot (x_1 x_4 + x_1 x_5 + x_1 x_6 + x_2 x_4 + x_2 x_5 + x_2 x_6 + x_3 x_4 + x_3 x_5 + x_3 x_6)] \\
+ & [(x_0 + jx_7 + j^2 x_8) \\
& \cdot (x_1 x_4 + j^2 x_1 x_5 + jx_1 x_6 + jx_2 x_4 + x_2 x_5 + j^2 x_2 x_6 + j^2 x_3 x_4 + jx_3 x_5 + x_3 x_6)] \\
+ & [(x_0 + j^2 x_7 + jx_8) \\
& \cdot (x_1 x_4 + jx_1 x_5 + j^2 x_1 x_6 + j^2 x_2 x_4 + x_2 x_5 + jx_2 x_6 + jx_3 x_4 + j^2 x_3 x_5 + x_3 x_6)] \\
= & \{3x_0[x_1 x_4 + x_2 x_5 + x_3 x_6]\} \\
+ & \{3x_7[x_1 x_5 + x_2 x_6 + x_3 x_4]\} \\
+ & \{3x_8[x_1 x_6 + x_2 x_4 + x_3 x_5]\}
\end{aligned}$$

Then we can define the norm of the ternary quaternion through the determinant

$$\begin{aligned}
Det \hat{Q} &= [(x_0 + jx_7 + j^2 x_8)(x_0 + j^2 x_7 + jx_8)(x_0 + x_7 + x_8)] \\
&+ [(x_1 + x_2 + x_3)(x_1 + jx_2 + j^2 x_3)(x_1 + j^2 x_2 + jx_3)] \\
&+ [(x_4 + x_5 + x_6)(x_4 + jx_5 + j^2 x_6)(x_4 + j^2 x_5 + jx_6)] \\
&- \{(x_0 + j^2 x_7 + jx_8)(x_1 + j^2 x_2 + jx_3)(x_4 + jx_5 + j^2 x_6) \\
&- (x_0 + jx_7 + j^2 x_8)(x_1 + jx_2 + j^2 x_3)(x_4 + j^2 x_5 + jx_6) \\
&- (x_0 + x_7 + x_8)(x_1 + x_2 + x_3)(x_4 + x_5 + x_6)\} \\
&= |z_0|^3 + |z_1|^3 + |z_2|^3 - (z_0 z_1 z_2 + \tilde{z}_0 \tilde{z}_1 \tilde{z}_2 + \tilde{\tilde{z}}_0 \tilde{\tilde{z}}_1 \tilde{\tilde{z}}_2)
\end{aligned} \tag{11}$$

$$\begin{aligned}
z_0 &= x_0 + x_7 q + x_8 q^2 & \tilde{z}_0 &= x_0 + jx_7 q + j^2 x_8 q^2 & \tilde{\tilde{z}}_0 &= x_0 + j^2 x_7 q + jx_8 q^2 \\
z_1 &= x_1 + x_2 q + x_3 q^2 & \tilde{z}_1 &= x_1 + jx_2 q + j^2 x_3 q^2 & \tilde{\tilde{z}}_1 &= x_1 + j^2 x_2 q + jx_3 q^2 \\
z_2 &= x_4 + x_5 q + x_6 q^2 & \tilde{z}_2 &= x_4 + jx_5 q + j^2 x_6 q^2 & \tilde{\tilde{z}}_2 &= x_4 + j^2 x_5 q + jx_6 q^2
\end{aligned} \tag{12}$$

or

$$\begin{aligned}
Det\hat{Q} &= [x_0^3 + x_7^3 + x_8^3 - 3x_0x_7x_8] \\
&+ [x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3] \\
&+ [(x_4^3 + x_5^3 + x_6^3 - 3x_4x_5x_6)] \\
&- \{(x_0 + j^2x_7 + jx_8) \\
&\cdot [x_1x_4 + jx_1x_5 + j^2x_1x_6 + j^2x_2x_4 + x_2x_5 + jx_2x_6 + jx_3x_4 + j^2x_3x_5 + x_3x_6]\} \\
&- \{(x_0 + x_7 + x_8) \\
&\cdot [x_1x_4 + j^2x_1x_5 + jx_1x_6 + jx_2x_4 + x_2x_5 + j^2x_2x_6 + j^2x_3x_4 + jx_3x_5 + x_3x_6]\} \\
&- \{(x_0 + jx_7 + j^2x_8) \\
&\cdot [x_1x_4 + x_1x_5 + x_1x_6 + x_2x_4 + x_2x_5 + x_2x_6 + x_3x_4 + x_3x_5 + x_3x_6]\}
\end{aligned} \tag{4}$$

$$\begin{aligned}
z_0 &= x_0 + x_7q + x_8q^2 & \tilde{z}_0 &= x_0 + jx_7q + j^2x_8q^2 & \tilde{\tilde{z}}_0 &= x_0 + j^2x_7q + jx_8q^2 \\
z_1 &= x_1 + x_2q + x_3q^2 & \tilde{z}_1 &= x_1 + jx_2q + j^2x_3q^2 & \tilde{\tilde{z}}_1 &= x_1 + j^2x_2q + jx_3q^2 \\
z_2 &= x_4 + x_5q + x_6q^2 & \tilde{z}_2 &= x_4 + jx_5q + j^2x_6q^2 & \tilde{\tilde{z}}_2 &= x_4 + j^2x_5q + jx_6q^2
\end{aligned} \tag{5}$$

or(1)

$$\begin{aligned}
Det\hat{Q} &= [x_0^3 + x_7^3 + x_8^3 - 3x_0x_7x_8] \\
&+ [x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3] \\
&+ [(x_4^3 + x_5^3 + x_6^3 - 3x_4x_5x_6)] \\
&- \{3x_0[x_1x_4 + x_2x_5 + x_3x_6]\} \\
&- \{3x_7[x_1x_5 + x_2x_6 + x_3x_4]\} \\
&- \{3x_8[x_1x_6 + x_2x_4 + x_3x_5]\}
\end{aligned} \tag{0}$$

or (2)

$$\begin{aligned}
Det\hat{Q} &= [x_0^3 + x_1^3 + x_4^3 - 3x_0x_1x_4] \\
&+ [x_7^3 + x_2^3 + x_6^3 - 3x_7x_2x_6] \\
&+ [(x_8^3 + x_5^3 + x_3^3 - 3x_8x_5x_3)] \\
&- \{3x_0[x_7x_8 + x_2x_5 + x_3x_6]\} \\
&- \{3x_1[x_2x_3 + x_5x_7 + x_6x_8]\} \\
&- \{3x_4[x_5x_6 + x_3x_7 + x_2x_8]\}
\end{aligned} \tag{-5}$$

$$\begin{aligned}
z_0 &= x_0 + x_1q + x_4q^2 & \tilde{z}_0 &= x_0 + jx_1q + j^2x_4^2 & \tilde{\tilde{z}}_0 &= x_0 + j^2x_1q + jx_4q^2 \\
z_1 &= x_7 + x_2q + x_6q^2 & \tilde{z}_1 &= x_7 + jx_2q + j^2x_6q^2 & \tilde{\tilde{z}}_1 &= x_7 + j^2x_2q + jx_6q^2 \\
z_2 &= x_8 + x_5q + x_3q^2 & \tilde{z}_2 &= x_8 + jx_5q + j^2x_3q^2 & \tilde{\tilde{z}}_2 &= x_8 + j^2x_5q + jx_3q^2
\end{aligned} \tag{-4}$$

or (3)

$$\begin{aligned}
Det\hat{Q} &= [x_0^3 + x_2^3 + x_5^3 - 3x_0x_2x_5] \\
&+ [x_7^3 + x_3^3 + x_4^3 - 3x_7x_3x_4] \\
&+ [(x_8^3 + x_1^3 + x_6^3 - 3x_8x_1x_6)] \\
&- \{3x_0[x_7x_8 + x_1x_4 + x_3x_6]\} \\
&- \{3x_2[x_1x_3 + x_6x_7 + x_4x_8]\} \\
&- \{3x_5[x_3x_4 + x_1x_7 + x_3x_8]\}
\end{aligned} \tag{-9}$$

$$\begin{aligned}
z_0 &= x_0 + x_2q + x_5q^2 & \tilde{z}_0 &= x_0 + jx_2q + j^2x_5q^2 & \tilde{\tilde{z}}_0 &= x_0 + j^2x_2q + jx_5q^2 \\
z_1 &= x_7 + x_3q + x_4q^2 & \tilde{z}_1 &= x_7 + jx_3q + j^2x_4q^2 & \tilde{\tilde{z}}_1 &= x_7 + j^2x_3q + jx_4q^2 \\
z_2 &= x_8 + x_1q + x_6q^2 & \tilde{z}_2 &= x_8 + jx_1q + j^2x_6q^2 & \tilde{\tilde{z}}_2 &= x_8 + j^2x_1q + jx_6q^2
\end{aligned} \tag{-8}$$

or (4)

$$\begin{aligned}
Det\hat{Q} &= [x_0^3 + x_3^3 + x_6^3 - 3x_0x_3x_6] \\
&+ [x_7^3 + x_1^3 + x_5^3 - 3x_1x_5x_7] \\
&+ [(x_8^3 + x_2^3 + x_4^3 - 3x_2x_4x_8)] \\
&- \{3x_0[x_7x_8 + x_1x_4 + x_2x_5]\} \\
&- \{3x_3[x_1x_2 + x_4x_7 + x_5x_8]\} \\
&- \{3x_6[x_4x_5 + x_2x_7 + x_1x_8]\}
\end{aligned} \tag{-13}$$

$$\begin{aligned}
z_0 &= x_0 + x_3q + x_6q^2 & \tilde{z}_0 &= x_0 + jx_3q + j^2x_6q^2 & \tilde{\tilde{z}}_0 &= x_0 + j^2x_3q + jx_6q^2 \\
z_1 &= x_7 + x_1q + x_5q^2 & \tilde{z}_1 &= x_7 + jx_1q + j^2x_5q^2 & \tilde{\tilde{z}}_1 &= x_7 + j^2x_1q + jx_5q^2 \\
z_2 &= x_8 + x_2q + x_4q^2 & \tilde{z}_2 &= x_8 + jx_2q + j^2x_4q^2 & \tilde{\tilde{z}}_2 &= x_8 + j^2x_2q + jx_4q^2
\end{aligned} \tag{-12}$$

$$\begin{array}{ccc}
0 & 8 & 7 \\
1 & 2 & 3 \\
4 & 6 & 5
\end{array}
\quad
\begin{array}{ccc}
0 & 8 & 7 \\
2 & 3 & 1 \\
5 & 4 & 6
\end{array}
\quad
\begin{array}{ccc}
0 & 8 & 7 \\
3 & 1 & 2 \\
6 & 5 & 4
\end{array} \tag{-11}$$

$$\begin{array}{ccc}
0 & 1 & 4 \\
2 & 7 & 6 \\
5 & 3 & 8
\end{array}
\quad
\begin{array}{ccc}
0 & 1 & 4 \\
6 & 2 & 7 \\
3 & 8 & 5
\end{array}
\quad
\begin{array}{ccc}
0 & 1 & 4 \\
7 & 6 & 2 \\
8 & 5 & 3
\end{array} \tag{-10}$$

$$\begin{array}{ccc}
0 & 2 & 5 \\
3 & 7 & 4 \\
6 & 1 & 8
\end{array}
\quad
\begin{array}{ccc}
0 & 2 & 5 \\
4 & 3 & 7 \\
1 & 8 & 6
\end{array}
\quad
\begin{array}{ccc}
0 & 2 & 5 \\
7 & 4 & 3 \\
8 & 6 & 1
\end{array}
\tag{-9}$$

$$\begin{array}{ccc}
0 & 3 & 6 \\
1 & 7 & 5 \\
4 & 2 & 8
\end{array}
\quad
\begin{array}{ccc}
0 & 3 & 6 \\
5 & 1 & 7 \\
2 & 8 & 4
\end{array}
\quad
\begin{array}{ccc}
0 & 3 & 6 \\
7 & 5 & 1 \\
8 & 4 & 2
\end{array}
\tag{-8}$$

One can see that this norm is a real number and if we define this norm to unit $Det\hat{Q} = 1$, it will define a cubic surface in $D = 9$.

11 Real ternary Tu3-algebra and root system

Let us give the link the nonions with the canonical $SU(3)$ matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{3}(q_1 + q_2 + q_3 + q_4 + jq_5 + q_6) \tag{-7}$$

$$\lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{i}{3}(-q_1 - q_2 - q_3 + q_4 + jq_5 + j^2q_6) \tag{-6}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \frac{1}{3}(q_1 + j^2q_2 + jq_3 + q_4 + jq_5 + j^2q_6) \tag{-5}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} = \frac{i}{3}(-q_1 - j^2q_2 - jq_3 + q_4 + jq_5 + j^2q_6) \tag{-4}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \frac{i}{3}(q_1 + jq_2 + j^2q_3 + q_4 + j^2q_5 + jq_6) \tag{-3}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} = \frac{i}{3}(q_1 + jq_2 + j^2q_3 - q_4 - j^2q_5 - jq_6) \tag{-2}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{(1-j)}(q_7 - jq_8) \quad (-1)$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \frac{-1}{\sqrt{3}}(jq_7 + j^2q_8) \quad (0)$$

Here the matrices $\lambda_i/2 = g_i$ satisfy to ordinary $SU(3)$ algebra:

$$[g_i, g_j]_{Z_2} = if_{ijk}g_k. \quad (0)$$

where f_{ijk} are completely antisymmetric and have the following values:

$$f_{123} = 1, f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2}, f_{458} = f_{678} = \frac{\sqrt{3}}{2}. \quad (0)$$

Now we introduce the plus-step operators:

$$Q_1 = Q_I^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad Q_2 = Q_{II}^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad Q_3 = Q_{III}^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad (1)$$

and on the minus-step operators:

$$Q_4 = Q_I^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad Q_5 = Q_{II}^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad Q_6 = Q_{III}^- = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (2)$$

We choose the following 3-diagonal operators :

$$Q_7 = Q_I^0 = \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\sqrt{\frac{2}{3}} \end{pmatrix} \quad Q_8 = Q_{II}^0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad Q_0 = Q_{III}^0 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}, \quad (3)$$

which produce the ternary Cartan subalgebra:

$$\{Q_0, Q_7, Q_8\} = 0. \quad (4)$$

The Q_k operators with $k = 0, 1, 2, \dots, 8$ satisfy to the following ternary S_3 commutation relations:

N	$\{klm\} \rightarrow \{n\}$	f_{klm}^n	N	$\{klm\} \rightarrow \{n\}$	f_{klm}^n	N	$\{klm\} \rightarrow \{n\}$	f_{klm}^n
1	$\{0, 1, 2\} \rightarrow \{6\}$	$\frac{1}{\sqrt{3}}$	29	$\{1, 2, 3\} \rightarrow \{0\}$	$\sqrt{3}$	57	$\{2, 4, 7\} \rightarrow \emptyset$	0
2	$\{0, 1, 3\} \rightarrow \{5\}$	$-\frac{1}{\sqrt{3}}$	30	$\{1, 2, 4\} \rightarrow \{2\}$	1	58	$\{2, 4, 8\} \rightarrow \emptyset$	0
3	$\{0, 1, 4\} \rightarrow \{0, 7, 8\}$	$\{0, 0, \sqrt{\frac{2}{3}}\}$	31	$\{1, 2, 5\} \rightarrow \{1\}$	1	59	$\{2, 5, 6\} \rightarrow \{6\}$	-1
4	$\{0, 1, 5\} \rightarrow \emptyset$	0	32	$\{1, 2, 6\} \rightarrow \emptyset$	0	60	$\{2, 5, 7\} \rightarrow \{0, 7, 8\}$	$\{\frac{3}{\sqrt{2}}, -1, -\frac{2}{\sqrt{3}}\}$
5	$\{0, 1, 6\} \rightarrow \emptyset$	0	33	$\{1, 2, 7\} \rightarrow \{6\}$	$-\sqrt{\frac{2}{3}}$	61	$\{2, 5, 8\} \rightarrow \{0, 7, 8\}$	$\{-\sqrt{\frac{3}{2}}, 0, 1\}$
6	$\{0, 1, 7\} \rightarrow \emptyset$	0	34	$\{1, 2, 8\} \rightarrow \{6\}$	$\sqrt{2}$	62	$\{2, 6, 7\} \rightarrow \emptyset$	0
7	$\{0, 1, 8\} \rightarrow \{1\}$	$-\sqrt{\frac{2}{3}}$	35	$\{1, 3, 4\} \rightarrow \{3\}$	-1	63	$\{2, 6, 8\} \rightarrow \emptyset$	0
8	$\{0, 2, 3\} \rightarrow \{4\}$	$\frac{1}{\sqrt{3}}$	36	$\{1, 3, 5\} \rightarrow \emptyset$	0	64	$\{2, 7, 8\} \rightarrow \{2\}$	$\frac{1}{\sqrt{3}}$
9	$\{0, 2, 4\} \rightarrow \emptyset$	0	37	$\{1, 3, 6\} \rightarrow \{1\}$	-1	65	$\{3, 4, 5\} \rightarrow \emptyset$	0
10	$\{0, 2, 5\} \rightarrow \{0, 7, 8\}$	$\{0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}\}$	38	$\{1, 3, 7\} \rightarrow \{5\}$	$\sqrt{\frac{2}{3}}$	66	$\{3, 4, 6\} \rightarrow \{4\}$	1
11	$\{0, 2, 6\} \rightarrow \emptyset$	0	39	$\{1, 3, 8\} \rightarrow \{5\}$	$\sqrt{2}$	67	$\{3, 4, 7\} \rightarrow \emptyset$	0
12	$\{0, 2, 7\} \rightarrow \{2\}$	$-\frac{1}{\sqrt{2}}$	40	$\{1, 4, 5\} \rightarrow \{5\}$	-1	68	$\{3, 4, 8\} \rightarrow \emptyset$	0
13	$\{0, 2, 8\} \rightarrow \{2\}$	$\frac{1}{\sqrt{6}}$	41	$\{1, 4, 6\} \rightarrow \{6\}$	1	69	$\{3, 5, 6\} \rightarrow \{5\}$	-1
14	$\{0, 3, 4\} \rightarrow \emptyset$	0	42	$\{1, 4, 7\} \rightarrow \{0, 7, 8\}$	$\{0, 0, \frac{1}{\sqrt{3}}\}$	70	$\{3, 5, 7\} \rightarrow \emptyset$	0
15	$\{0, 3, 5\} \rightarrow \emptyset$	0	43	$\{1, 4, 8\} \rightarrow \{0, 7, 8\}$	$\{\sqrt{6}, \sqrt{3}, 0\}$	71	$\{3, 5, 8\} \rightarrow \emptyset$	0
16	$\{0, 3, 6\} \rightarrow \{0, 7, 8\}$	$\{0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}\}$	44	$\{1, 5, 6\} \rightarrow \emptyset$	0	72	$\{3, 6, 7\} \rightarrow \{0, 7, 8\}$	$\{-\frac{3}{\sqrt{2}}, 1, -\frac{2}{\sqrt{3}}\}$
17	$\{0, 3, 7\} \rightarrow \{3\}$	$\frac{1}{\sqrt{2}}$	45	$\{1, 5, 7\} \rightarrow \emptyset$	0	73	$\{3, 6, 8\} \rightarrow \{0, 7, 8\}$	$\{-\sqrt{\frac{3}{2}}, 0, -1\}$
18	$\{0, 3, 8\} \rightarrow \{3\}$	$\frac{1}{\sqrt{6}}$	46	$\{1, 5, 8\} \rightarrow \emptyset$	0	74	$\{3, 7, 8\} \rightarrow \{3\}$	$\frac{1}{\sqrt{3}}$
19	$\{0, 4, 5\} \rightarrow \{3\}$	$-\frac{1}{\sqrt{3}}$	47	$\{1, 6, 7\} \rightarrow \emptyset$	0	75	$\{4, 5, 6\} \rightarrow \{0\}$	$-\sqrt{3}$
20	$\{0, 4, 6\} \rightarrow \{2\}$	$\frac{1}{\sqrt{3}}$	48	$\{1, 6, 8\} \rightarrow \emptyset$	0	76	$\{4, 5, 7\} \rightarrow \{3\}$	$\sqrt{\frac{2}{3}}$
21	$\{0, 4, 7\} \rightarrow \emptyset$	0	49	$\{1, 7, 8\} \rightarrow \{1\}$	$\frac{1}{\sqrt{3}}$	77	$\{4, 5, 8\} \rightarrow \{3\}$	$-\sqrt{2}$
22	$\{0, 4, 8\} \rightarrow \{4\}$	$\sqrt{\frac{2}{3}}$	50	$\{2, 3, 4\} \rightarrow \emptyset$	0	78	$\{4, 6, 7\} \rightarrow \{2\}$	$-\sqrt{\frac{2}{3}}$
23	$\{0, 5, 6\} \rightarrow \{1\}$	$-\frac{1}{\sqrt{3}}$	51	$\{2, 3, 5\} \rightarrow \{3\}$	1	79	$\{4, 6, 8\} \rightarrow \{2\}$	$-\sqrt{2}$
24	$\{0, 5, 7\} \rightarrow \{5\}$	$\frac{1}{\sqrt{2}}$	52	$\{2, 3, 6\} \rightarrow \{2\}$	1	80	$\{4, 7, 8\} \rightarrow \{4\}$	$-\frac{1}{\sqrt{3}}$
25	$\{0, 5, 8\} \rightarrow \{5\}$	$-\frac{1}{\sqrt{6}}$	53	$\{2, 3, 7\} \rightarrow \{4\}$	$2\sqrt{\frac{2}{3}}$	81	$\{5, 6, 7\} \rightarrow \{1\}$	$-2\sqrt{\frac{2}{3}}$
26	$\{0, 6, 7\} \rightarrow \{6\}$	$-\frac{1}{\sqrt{2}}$	54	$\{2, 3, 8\} \rightarrow \emptyset$	0	82	$\{5, 6, 8\} \rightarrow \emptyset$	0
27	$\{0, 6, 8\} \rightarrow \{6\}$	$-\frac{1}{\sqrt{6}}$	55	$\{2, 4, 5\} \rightarrow \{4\}$	-1	83	$\{5, 7, 8\} \rightarrow \{5\}$	$-\frac{1}{\sqrt{3}}$
28	$\{0, 7, 8\} \rightarrow \emptyset$	0	56	$\{2, 4, 6\} \rightarrow \emptyset$	0	84	$\{6, 7, 8\} \rightarrow \{6\}$	$-\frac{1}{\sqrt{3}}$

We have got 84 commutations relations. One commutation relation, $\{Q_0, Q_7, Q_8\}_{S_3} = 0$, provide the Cartan subalgebra. Let separate the rest 83 commutation relations on the 5 groups(18+18+27+18+2). The first group contains itself the following 18 commutation relations in one group (see Table):

Table 4: *I-The root system of the step operators*

$\{klm\} \rightarrow \{n\}$	f_{klm}^n	$\{klm\} \rightarrow \{n\}$	f_{klm}^n	$\{klm\} \rightarrow \{n\}$	f_{klm}^n	$\vec{\alpha}_i$
$\{1, 7, 8\} \rightarrow \{1\}$	$\frac{1}{\sqrt{3}}$	$\{2, 7, 8\} \rightarrow \{2\}$	$\frac{1}{\sqrt{3}}$	$\{3, 7, 8\} \rightarrow \{3\}$	$\frac{1}{\sqrt{3}}$	$\vec{\alpha}_1$
$\{4, 7, 8\} \rightarrow \{4\}$	$-\frac{1}{\sqrt{3}}$	$\{5, 7, 8\} \rightarrow \{5\}$	$-\frac{1}{\sqrt{3}}$	$\{6, 7, 8\} \rightarrow \{6\}$	$-\frac{1}{\sqrt{3}}$	$\vec{\alpha}_4$
$\{0, 1, 7\} \rightarrow \emptyset$	0	$\{0, 2, 7\} \rightarrow \{2\}$	$-\frac{1}{\sqrt{2}}$	$\{0, 3, 7\} \rightarrow \{3\}$	$\frac{1}{\sqrt{2}}$	$\vec{\alpha}_2$
$\{0, 4, 7\} \rightarrow \emptyset$	0	$\{0, 5, 7\} \rightarrow \{5\}$	$\frac{1}{\sqrt{2}}$	$\{0, 6, 7\} \rightarrow \{6\}$	$-\frac{1}{\sqrt{2}}$	$\vec{\alpha}_5$
$\{0, 1, 8\} \rightarrow \{1\}$	$-\sqrt{\frac{2}{3}}$	$\{0, 2, 8\} \rightarrow \{2\}$	$\frac{1}{\sqrt{6}}$	$\{0, 3, 8\} \rightarrow \{3\}$	$\frac{1}{\sqrt{6}}$	$\vec{\alpha}_3$
$\{0, 4, 8\} \rightarrow \{4\}$	$\sqrt{\frac{2}{3}}$	$\{0, 5, 8\} \rightarrow \{5\}$	$-\frac{1}{\sqrt{6}}$	$\{0, 6, 8\} \rightarrow \{6\}$	$-\frac{1}{\sqrt{6}}$	$\vec{\alpha}_6$

$$\begin{aligned}
\{\vec{H}_\alpha, Q_1\}_{S_3} &= \vec{\alpha}_1 Q_1, & \{\vec{H}_\alpha, Q_4\}_{S_3} &= \vec{\alpha}_4 Q_4, \\
\{\vec{H}_\alpha, Q_2\}_{S_3} &= \vec{\alpha}_2 Q_2, & \{\vec{H}_\alpha, Q_5\}_{S_3} &= \vec{\alpha}_5 Q_5, \\
\{\vec{H}_\alpha, Q_3\}_{S_3} &= \vec{\alpha}_3 Q_3, & \{\vec{H}_\alpha, Q_6\}_{S_3} &= \vec{\alpha}_6 Q_6
\end{aligned} \tag{2}$$

where $\vec{H}_\alpha = \{H_1, H_2, H_3\} = \{(Q_7, Q_8), (Q_0, Q_7), (Q_0, Q_8)\}$ and

$$\begin{aligned}
\vec{\alpha}_1 &= -\vec{\alpha}_4 = \left\{ \frac{1}{\sqrt{3}}, 0, -\sqrt{\frac{2}{3}} \right\} \\
\vec{\alpha}_2 &= -\vec{\alpha}_5 = \left\{ \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}} \right\} \\
\vec{\alpha}_3 &= -\vec{\alpha}_6 = \left\{ \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}} \right\},
\end{aligned} \tag{0}$$

where

$$\begin{aligned}
\langle \vec{\alpha}_i, \vec{\alpha}_i \rangle &= 1, & i &= 1, 2, \dots, 6, \\
\langle \vec{\alpha}_i, \vec{\alpha}_j \rangle &= 0, & i, j &= 1, 2, \dots, 6, & i \neq j.
\end{aligned} \tag{-1}$$

Note, that

$$\begin{aligned}
\langle \vec{\alpha}_i^0, \vec{\alpha}_i^0 \rangle &= \frac{2}{3}, & i &= 1, 2, \dots, 6, \\
\vec{\alpha}_1^0 + \vec{\alpha}_2^0 + \vec{\alpha}_3^0 &= 0, \\
\langle \vec{\alpha}_1^0, \vec{\alpha}_2^0 \rangle &= \langle \vec{\alpha}_2^0, \vec{\alpha}_3^0 \rangle = \langle \vec{\alpha}_3^0, \vec{\alpha}_1^0 \rangle = -\frac{1}{3},
\end{aligned} \tag{-3}$$

Table 5: *II-The dual roots*

$\{klm\} \rightarrow \{n\}$	f_{klm}^n	$\{klm\} \rightarrow \{n\}$	f_{klm}^n	$\{klm\} \rightarrow \{n\}$	f_{klm}^n	$\vec{\beta}_i$
$\{0, 1, 2\} \rightarrow \{6\}$	$\frac{1}{\sqrt{3}}$	$\{7, 1, 2\} \rightarrow \{6\}$	$-\frac{\sqrt{2}}{\sqrt{3}}$	$\{8, 1, 2\} \rightarrow \{6\}$	$\sqrt{2}$	$\vec{\beta}_1$
$\{0, 4, 5\} \rightarrow \{3\}$	$-\frac{1}{\sqrt{3}}$	$\{7, 4, 5\} \rightarrow \{3\}$	$\frac{\sqrt{2}}{\sqrt{3}}$	$\{8, 4, 5\} \rightarrow \{3\}$	$-\sqrt{2}$	$\vec{\beta}_4$
$\{0, 2, 3\} \rightarrow \{4\}$	$\frac{1}{\sqrt{3}}$	$\{7, 2, 3\} \rightarrow \{4\}$	$\frac{2\sqrt{2}}{\sqrt{3}}$	$\{8, 2, 3\} \rightarrow \{\emptyset\}$	0	$\vec{\beta}_2$
$\{0, 5, 6\} \rightarrow \{1\}$	$-\frac{1}{\sqrt{3}}$	$\{7, 5, 6\} \rightarrow \{1\}$	$-\frac{2\sqrt{2}}{\sqrt{3}}$	$\{8, 5, 6\} \rightarrow \{\emptyset\}$	0	$\vec{\beta}_5$
$\{0, 3, 1\} \rightarrow \{5\}$	$\frac{1}{\sqrt{3}}$	$\{7, 3, 1\} \rightarrow \{5\}$	$-\frac{\sqrt{2}}{\sqrt{3}}$	$\{8, 3, 1\} \rightarrow \{5\}$	$-\sqrt{2}$	$\vec{\beta}_3$
$\{0, 6, 4\} \rightarrow \{2\}$	$-\frac{1}{\sqrt{3}}$	$\{7, 6, 4\} \rightarrow \{2\}$	$\frac{\sqrt{2}}{\sqrt{3}}$	$\{8, 6, 4\} \rightarrow \{2\}$	$\sqrt{2}$	$\vec{\beta}_6$

where

$$\vec{\alpha}_i^0 = \{0, \vec{\alpha}_{i2}, \vec{\alpha}_{i3}\}, \quad i = 1, 2, \dots, 6, \quad (-2)$$

are the binary non-zero roots, in which the first components $\vec{\alpha}_{i1}$, $i = 1, \dots, 6$, are equal zero. Thus, we have got

$$\frac{\langle \vec{\alpha}_i, \vec{\alpha}_i \rangle}{\langle \vec{\alpha}_i^0, \vec{\alpha}_i^0 \rangle} = \frac{3}{2} \quad (-1)$$

Note, that there is only one simple root, since all α_i , $i = 1, 2, 3$ or $i = 4, 5, 6$, are related by usual Z_2 transformations, $\vec{\alpha}_i = -\vec{\alpha}_{i+3}$, ($i = 1, 2, 3$), or by Z_3 transformations:

$$\vec{\alpha}_2 = R^V(q)\vec{\alpha}_1 = O(2\pi/3)\vec{\alpha}_1, \quad \alpha_3 = R^V(q)^2\vec{\alpha}_1 = O(4\pi/3)\vec{\alpha}_1, \quad (0)$$

where

$$R^V(q) = O(2\pi/3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & -1/2 \end{pmatrix}, \quad (1)$$

$$R^V(q^2) = (R^V(q))^2 \text{ and } (R^V(q))^3 = R^V(q_0)$$

Now one can unify in second group the other 18 commutations relations:

Using the properties of multiplications:

$$\begin{aligned} Q_6 &= Q_1 Q_2, Q_4 = Q_2 Q_3, Q_5 = Q_3 Q_1, \\ Q_3 &= Q_4 Q_5, Q_1 = Q_5 Q_6, Q_2 = Q_6 Q_4, \end{aligned} \quad (0)$$

one can introduce the new systems of the *beta*-roots (see Table II):

Table 6: *III-The root system of the step operators*

$\{klm\} \rightarrow \{n\}$	f_{klm}^n	$\{klm\} \rightarrow \{n\}$	f_{klm}^n	$\{klm\} \rightarrow \{n\}$	f_{klm}^n
$\{1, 4, 0\} \rightarrow \{0, 7, 8\}$	$\{0, 0, \frac{\sqrt{2}}{\sqrt{3}}\}$	$\{1, 5, 0\} \rightarrow \emptyset$	$\{0, 0, 0\}$	$\{1, 6, 0\} \rightarrow \emptyset$	$\{0, 0, 0\}$
$\{1, 4, 7\} \rightarrow \{0, 7, 8\}$	$\{0, 0, \frac{1}{\sqrt{3}}\}$	$\{1, 5, 7\} \rightarrow \emptyset$	$\{0, 0, 0\}$	$\{1, 6, 7\} \rightarrow \emptyset$	$\{0, 0, 0\}$
$\{1, 4, 8\} \rightarrow \{0, 7, 8\}$	$\{\sqrt{6}, \sqrt{3}, 0\}$	$\{1, 5, 8\} \rightarrow \emptyset$	$\{0, 0, 0\}$	$\{1, 6, 8\} \rightarrow \emptyset$	$\{0, 0, 0\}$
$\{2, 4, 0\} \rightarrow \emptyset$	$\{0, 0, 0\}$	$\{2, 5, 0\} \rightarrow \{0, 7, 8\}$	$\{0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}\}$	$\{2, 6, 0\} \rightarrow \emptyset$	$\{0, 0, 0\}$
$\{2, 4, 7\} \rightarrow \emptyset$	$\{0, 0, 0\}$	$\{2, 5, 7\} \rightarrow \{0, 7, 8\}$	$\{\frac{3}{\sqrt{2}}, -1, -\frac{2}{\sqrt{3}}\}$	$\{2, 6, 7\} \rightarrow \emptyset$	$\{0, 0, 0\}$
$\{2, 4, 8\} \rightarrow \emptyset$	$\{0, 0, 0\}$	$\{2, 5, 8\} \rightarrow \{0, 7, 8\}$	$\{-\frac{\sqrt{3}}{\sqrt{2}}, 0, 1\}$	$\{2, 6, 8\} \rightarrow \emptyset$	$\{0, 0, 0\}$
$\{3, 4, 0\} \rightarrow \emptyset$	$\{0, 0, 0\}$	$\{3, 5, 0\} \rightarrow \emptyset$	$\{0, 0, 0\}$	$\{3, 6, 0\} \rightarrow \{0, 7, 8\}$	$\{0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}\}$
$\{3, 4, 7\} \rightarrow \emptyset$	$\{0, 0, 0\}$	$\{3, 5, 7\} \rightarrow \emptyset$	$\{0, 0, 0\}$	$\{3, 6, 7\} \rightarrow \{0, 7, 8\}$	$\{-\frac{3}{\sqrt{2}}, 1, -\frac{2}{\sqrt{3}}\}$
$\{3, 4, 8\} \rightarrow \emptyset$	$\{0, 0, 0\}$	$\{3, 5, 8\} \rightarrow \emptyset$	$\{0, 0, 0\}$	$\{3, 6, 8\} \rightarrow \{0, 7, 8\}$	$\{-\sqrt{\frac{3}{2}}, 0, -1\}$

$$\begin{aligned}
\vec{\beta}_1 = -\vec{\beta}_4 &= \left\{ \frac{1}{\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}}, \sqrt{2} \right\} \\
\vec{\beta}_2 = -\vec{\beta}_5 &= \left\{ \frac{1}{\sqrt{3}}, \frac{2\sqrt{2}}{\sqrt{3}}, 0 \right\} \\
\vec{\beta}_3 = -\vec{\beta}_6 &= \left\{ \frac{1}{\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}}, -\sqrt{2} \right\}
\end{aligned} \tag{-2}$$

$$\begin{aligned}
\langle \vec{\beta}_i, \vec{\beta}_i \rangle &= 3, \quad i = 1, 2, \dots, 6, \\
\langle \vec{\beta}_i, \vec{\beta}_j \rangle &= -1, \quad i, j = 1, 2, 3, \quad i \neq j.
\end{aligned} \tag{-3}$$

Note, that there is also only one simple dual root, since all β_i , $i = 1, 2, 3$ or $i = 4, 5, 6$, are related by usual Z_2 transformations, $\vec{\beta}_i = -\vec{\beta}_{i+3}$, ($i = 1, 2, 3$), or by Z_3 transformations:

$$\vec{\beta}_2 = R^V(q)\vec{\beta}_1 = O(2\pi/3)\vec{\beta}_1, \quad \beta_3 = R^V(q)^2\vec{\beta}_1 = O(4\pi/3)\vec{\beta}_1, \tag{-2}$$

where

$$R^V(q) = O(2\pi/3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & -1/2 \end{pmatrix}, \tag{-1}$$

$R^V(q^2) = (R^V(q))^2$ and $(R^V(q))^3 = R^V(q_0)$

The third group contains itself 27 commutation relations amonth them there are only 9 have the non-zero results:

The fourth group has also the 18 commutations relations:

The last, 5-th , group has only two but very important commutation relations:

Table 7: *IV-The root system of the step operators*

$\{klm\} \rightarrow \{n\}$	f_{klm}^n	$\{klm\} \rightarrow \{n\}$	f_{klm}^n	$\{klm\} \rightarrow \{n\}$	f_{klm}^n
$\{1, 2, 4\} \rightarrow \{2\}$	$\{1\}$	$\{1, 2, 5\} \rightarrow \{1\}$	$\{1\}$	$\{1, 2, 6\} \rightarrow \{\emptyset\}$	$\{0\}$
$\{1, 3, 4\} \rightarrow \{3\}$	$\{-1\}$	$\{1, 3, 5\} \rightarrow \{\emptyset\}$	$\{0\}$	$\{1, 3, 6\} \rightarrow \{1\}$	$\{-1\}$
$\{2, 3, 4\} \rightarrow \{\emptyset\}$	$\{0\}$	$\{2, 3, 5\} \rightarrow \{3\}$	$\{1\}$	$\{2, 3, 6\} \rightarrow \{2\}$	$\{1\}$
$\{1, 4, 5\} \rightarrow \{5\}$	$\{-1\}$	$\{2, 4, 5\} \rightarrow \{4\}$	$\{-1\}$	$\{3, 4, 5\} \rightarrow \{\emptyset\}$	$\{0\}$
$\{1, 4, 6\} \rightarrow \{6\}$	$\{1\}$	$\{2, 4, 6\} \rightarrow \{\emptyset\}$	$\{0\}$	$\{3, 4, 6\} \rightarrow \{4\}$	$\{1\}$
$\{1, 5, 6\} \rightarrow \{\emptyset\}$	$\{0\}$	$\{2, 5, 6\} \rightarrow \{6\}$	$\{-1\}$	$\{3, 5, 6\} \rightarrow \{5\}$	$\{-1\}$

Table 8: *V-The root system of the step operators*

$\{klm\} \rightarrow \{n\}$	f_{klm}^n	$\{klm\} \rightarrow \{n\}$	f_{klm}^n
$\{1, 2, 3\} \rightarrow \{0\}$	$\{\sqrt{3}\}$	$\{4, 5, 6\} \rightarrow \{0\}$	$\{-\sqrt{3}\}$

12 \mathbb{C}_N - Clifford algebra

We begin with a V , a finite-dimensional vector space over the fields, $\Lambda = \mathbb{R}, \mathbb{C}$ or $\Lambda = \mathbb{TC}$.

We introduce the tensor algebra $T(V) = \bigoplus_{n \geq 0} \otimes^n V$, with $\otimes^0 V = \Lambda$.

The product in $T(V)$ one can define as follows: $v_1 \otimes \dots \otimes v_p \in V^{\otimes p}$ and $u_1 \otimes \dots \otimes u_q \in V^{\otimes q}$, then their product is $v_1 \otimes \dots \otimes v_p \otimes u_1 \otimes \dots \otimes u_q \in V^{\otimes(p+q)}$. For example, if V has a basis $\{x, y\}$, then $T(V)$ has a basis $\{1, x, y, xy, yx, x^2, y^2, x^2y, y^2x, x^2y^2, \dots\}$. Suppose now we introduce into V a trilinear form (\dots, \dots, \dots) . Let $J = \langle v \otimes v \otimes v - (v, v, v) \cdot 1 | v \in V \rangle$ an ideal in $T(V)$ and put

$$TCl(V) = T(V)/J, \quad (0)$$

the Clifford algebra over V with trilinear form (\dots, \dots, \dots) .

To generalize the binary Clifford algebra one can introduce the following generators q_1, q_2, \dots, q_n and relations:

$$q_k^3 = 1 \quad (1)$$

and

$$q_k q_l = j q_l q_k, \quad q_l q_k = j^2 q_k q_l, \quad n \geq l > k \geq 1, \quad (2)$$

where $j = \exp(2\pi/3)$. One can immediately find two types of the S_3 identities. The first type of such identities are:

$$\begin{aligned} q_k q_l q_k + q_k^2 q_l + q_l q_k^2 &= (j + 1 + j^2) q_k^2 q_l = 0, \\ q_k q_l q_k + j^2 q_k^2 q_l + j q_l q_k^2 &= (j + j^2 + 1) q_k^2 q_l = 0, \\ q_k q_l q_k + j q_k^2 q_l + j^2 q_l q_k^2 &= (3j) q_k^2 q_l, \end{aligned} \quad (0)$$

or

$$\begin{aligned}
q_l q_k q_l + q_1^2 q_k + q_k q_l^2 &= (j^2 + j + 1) q_k q_l^2 = 0, \\
q_l q_k q_l + j^2 q_1^2 q_k + j q_k q_l^2 &= (j^2 + 1 + j) q_k q_l^2 = 0, \\
q_l q_k q_l + j q_1^2 q_k + j^2 q_k q_l^2 &= (3j^2) q_k q_l^2,
\end{aligned} \tag{-2}$$

The second type of the identities relate to the triple product of the generators with all different indexes, for example , one can take take $n \geq m > l > k \geq 1$. Then one can easily get :

$$\begin{aligned}
& (q_k q_l q_m + q_l q_m q_k + q_m q_k q_l) + (q_m q_l q_k + q_l q_k q_m + q_k q_m q_l) \\
&= \left((1 + j^2 + j^2) + (1 + j + j) \right) q_k q_l q_m = \left((j^2 - j) + (j - j^2) \right) q_k q_l q_m \\
&= \left((1 + j + j) + (1 + j^2 + j^2) \right) q_m q_l q_k = \left((j - j^2) + (j^2 - j) \right) q_m q_l q_k \\
&= 0.
\end{aligned} \tag{-5}$$

From these two types of the identities one can see, that S_{3+} -symmetric sum

$$\sum_{S_{3+}} = q_k q_l q_m + q_l q_m q_k + q_m q_k q_l + q_m q_l q_k + q_l q_k q_m + q_k q_m q_l = \{q_k q_l q_m\} \tag{-4}$$

is not equal zero just in one case, when all indexes, k, l, m are equal, *i.e.*:

$$\sum_{S_{3+}} (q_k q_l q_m) = q_k q_l q_m + q_l q_m q_k + q_m q_k q_l + q_m q_l q_k + q_l q_k q_m + q_k q_m q_l = 6\delta_{klm}. \tag{-3}$$

$$\sum_{k=1}^{k=n} \sum_{l=1}^{l=n} \sum_{m=1}^{m=n} (q_k q_l q_m) = \sum_{S_{3+}} (q_k q_l q_m) = (q_k q_l q_m + q_l q_m q_k + q_m q_k q_l + q_m q_l q_k + q_l q_k q_m + q_k q_m q_l) = n\delta_{klm}. \tag{-2}$$

$TCl(V)$ is a Z_3 -graded algebra. We put

$$T(V)_0 = \oplus_{n=3k} V^{\otimes n}, \quad T(V)_1 = \oplus_{n=3k+1} V^{\otimes n}, \quad T(V)_2 = \oplus_{n=3k+2} V^{\otimes n}. \tag{-1}$$

Also, one can see

$$TCl(V)_0 = TCl(V)_0 \oplus TCl(V)_1 \oplus TCl(V)_2, \quad TCl(V)_k = T(V)_k / J_k. \tag{0}$$

$$J_k = J \bigcap T(V)_k. \tag{1}$$

If $\dim_{\Lambda} V = n$ and $\{q_1, \dots, q_n\}$ is an ortghonal basis for V with $(q_k, q_l, q_m) = \lambda_k \delta_{k,l,m}$, then the dimension $\dim_{\Lambda} TCl(V) = 3^n$ and $\{\prod q_k^{l_k}\}$ is a basis where l_k is 0,1, or 2.

$n - gen$	1	2	3	4	5	6
TCl_0	1	1 + 2	1 + 7 + 1	1 + 16 + 10	1 + 30 + 45 + 5	1 + 50 + 141 + 50 + 1
TCl_1	1	2 + 1	3 + 6	4 + 19 + 4	5 + 45 + 30 + 1	—
TCl_2	1	3	6 + 3	1 + 16 + 1	15 + 51 + 15	—
Σ	3	$3 \times 3 = 9$	9×3	81	$81 \times 3 = 243$	$243 \times 3 = 729$

(2)

0.	1	1
1.	$< q_1, q_2, q_3, q_4 >$	4
2.	$< q_1^2, q_2^2, q_3^2, q_4^2 >$	—
	$< q_1 q_2, q_1 q_3, q_1 q_4, q_2 q_3, q_2 q_4, q_3 q_4 >$	10
3.	$< q_1^2 q_2, q_1^2 q_3, q_1^2 q_4, q_2^2 q_1, q_2^2 q_3, q_2^2 q_4, q_3^2 q_1, q_3^2 q_2, q_3^2 q_4, q_4^2 q_1, q_4^2 q_2, q_4^2 q_3 >$	—
	$< q_1 q_2 q_3, q_1 q_2 q_4, q_1 q_3 q_4, q_2 q_3 q_4 >$	16
4.	$< q_1^2 q_2^2, q_1^2 q_3^2, q_1^2 q_4^2, q_2^2 q_3^2, q_2^2 q_4^2, q_3^2 q_4^2 >$	—
	$< q_1^2 q_2 q_3, q_1^2 q_2 q_4, q_1^2 q_3 q_4, q_2^2 q_1 q_3, q_2^2 q_1 q_4, q_2^2 q_3 q_4, q_3^2 q_1 q_3, q_3^2 q_1 q_4, q_3^2 q_3 q_4, q_4^2 q_1 q_2, q_4^2 q_1 q_3, q_4^2 q_2 q_3 >$	—
	$< q_1 q_2 q_3 q_4 >$	19
5.	$< q_1^2 q_2^2 q_3, q_1^2 q_2^2 q_4, q_1^2 q_3^2 q_2, q_1^2 q_3^2 q_4, q_1^2 q_4^2 q_2, q_1^2 q_4^2 q_3, q_2^2 q_3^2 q_1, q_2^2 q_3^2 q_4, q_2^2 q_4^2 q_1, q_2^2 q_4^2 q_3, q_3^2 q_4^2 q_1, q_3^2 q_4^2 q_2 >$	—
	$< q_1^2 q_2 q_3 q_4, q_2^2 q_1 q_3 q_4, q_3^2 q_1 q_2 q_4, q_4^2 q_1 q_2 q_3 >$	16
6.	$< q_1^2 q_2^2 q_3^2, q_1^2 q_2^2 q_4^2, q_1^2 q_3^2 q_4^2, q_2^2 q_3^2 q_4^2 >$	—
	$< q_1^2 q_2^2 q_3 q_4, q_1^2 q_3^2 q_2 q_4, q_1^2 q_4^2 q_2 q_3, q_2^2 q_3^2 q_1 q_4, q_2^2 q_4^2 q_1 q_3, q_3^2 q_4^2 q_1 q_2 >$	10
7.	$< q_1^2 q_2^2 q_3^2 q_4, q_2^2 q_3^2 q_4^2 q_1, q_3^2 q_4^2 q_1^2 q_2, q_4^2 q_1^2 q_2^2 q_3 >$	4
8.	$< q_1^2 q_2^2 q_3^2 q_4^2 >$	1

(3)

Solutions of the Phiafagor Equations for 3D case

a	b	c	d	d^3
2	3	3	2	8
2	3	4	3	27
3	19	27	28	21952
3	31	38	42	74088
4	6	6	4	64
4	6	8	6	216
5	25	42	42	74088
6	9	9	6	216
6	9	12	9	729

(4)

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